A Derivation of Maxwell Equations in Quaternion Space

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Quaternion space and its respective Quaternion Relativity (it also may be called as Rotational Relativity) have been defined in a number of papers including [1], and this new theory is capable to describe relativistic motion in elegant and straightforward way. Nonetheless there are subsequent theoretical developments which remain an open question, for instance how to derive Maxwell equations in Q-space. The purpose of the present paper is to derive a consistent description of Maxwell equations in Q-space. First we consider a simplified method similar to the Feynman's derivation of Maxwell equations from Lorentz force. And then we present another derivation method using Dirac decomposition, introduced by Gersten (1999). Further observation is of course recommended in order to refute or verify some implication of this proposition.

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1 Introduction

Quaternion space and its respective Quaternion Relativity (it also may be called as Rotational Relativity) has been defined in a number of papers including [1], and it can be shown that this new theory is capable to describe relativistic motion in elegant and straightforward way. For instance, it can be shown that the Pioneer spacecraft's Doppler shift anomaly can be explained as a relativistic effect of Quaternion Space [11]. The Yang-Mills field also can be shown to be consistent with Quaternion Space [1]. Nonetheless there are subsequent theoretical developments which remain an open issue, for instance to derive Maxwell equations in Q-space [1].

The purpose of the present article is to derive a consistent description of Maxwell equations in Q-space. First we consider a simplified method similar to the Feynman's derivation of Maxwell equations from Lorentz force. And then we present another method using Dirac decomposition, introduced by Gersten (1999). In the first section we will shortly review the basics of Quaternion space as introduced in [1].

Further observation is of course recommended in order to verify or refute the propositions outlined herein.

2 Basic definitions of Q-relativity physics

In this section, we will review some basic definitions of quaternion number and then discuss their implications to quaternion relativity (Q-relativity) physics [1].

Quaternion number belongs to the group of “very good” algebras: of real, complex, quaternion, and octonion [1], and it is often defined as follows [1]:

$$Q \equiv a + bi + cj + dk. \tag{1}$$

Where a,b,c,d are real numbers, and i,j,k are imaginary quaternion units. These Q-units can be represented either via 2x2 matrices or 4x4 matrices. There is quaternionic multiplication rule which acquires compact form [1]:

$$1q_k = q_k 1 = q_k, \quad q_j q_k = -\delta_{jk} + \varepsilon_{jkn} q_n, \tag{2}$$

Where $\delta_{kn}$and ε_{jkn}represent 3-dimensional symbols of Kronecker and Levi-Civita, respectively.

In the context of Quaternion Space [1], it is also possible to write the dynamics equations of classical mechanics for an inertial observer in constant Q-basis. SO(3,R)-invariance of two vectors allow to represent these dynamics equations in Q-vector form [1]:

$$m\frac{d^2}{dt^2}\left(x_k q_k\right) = F_k q_k. \tag{3}$$

Because of antisymmetry of the connection (generalised angular velocity) the dynamics equations can be written in vector components, by conventional vector notation [1]:

$$m(\vec{a} + 2\vec{\Omega} \times \vec{v} + \vec{\Omega} \times \vec{r} + \vec{\Omega} \times \left(\vec{\Omega} \times \vec{r}\right)) = \vec{F} \tag{4}$$

Therefore, from equation (4) one recognizes known types of classical acceleration, i.e. linear, coriolis, angular, centripetal.

From this viewpoint one may consider a generalization of Minkowski metric interval into biquaternion form [1]:

$$dz = \left(dx_k + idt_k\right) q_k, \tag{5}$$

With some novel properties, i.e.:

- temporal interval is defined by imaginary vector;
- space-time of the model appears to have six dimensions (6D);

- vector of the displacement of the particle and vector of corresponding time change must always be normal to each other, or:

$$dx_k dt_k = 0. \tag{6}$$

One advantage of this Quaternion Space representation is that it enables to describe rotational motion with great clarity.

After this short review of Q-space, next we will discuss a simplified method to derive Maxwell equations from Lorentz force, in a similar way with Feynman's derivation method using commutative relation [2][10].

3 An intuitive approach from Feynman's derivation method

A simplified derivation of Maxwell equations will be discussed here using a similar approach with Feynman's derivation [2][3][10].

We can introduce now the Lorentz force into equation (4), to become:

$$m(\frac{d\vec{v}}{dt} + 2\vec{\Omega} \times \vec{v} + \vec{\Omega} \times \vec{r} + \vec{\Omega} \times \left(\vec{\Omega} \times \vec{r}\right)) = q_{\otimes}(\vec{E} + \frac{1}{c}\vec{v}x\vec{B}), \tag{7}$$

Or

$$(\frac{d\vec{v}}{dt}) = \frac{q_{\otimes}}{m}(\vec{E} + \frac{1}{c}\vec{v}x\vec{B}) - 2\vec{\Omega} \times \vec{v} - \vec{\Omega} \times \vec{r} - \vec{\Omega} \times \left(\vec{\Omega} \times \vec{r}\right). \tag{8}$$

We note here that q variable here denotes electric charge, not quaternion basis.

Interestingly, equation (4) can be compared directly to equation (8) in [2]:

$$m\ddot{x} = F - m\vec{r} \times \vec{\Omega} - m2\dot{x} \times \vec{\Omega} - m\vec{\Omega} \times \left(\vec{r} \times \vec{\Omega}\right)), \tag{9}$$

In other words, we find an exact correspondence between quaternion version of Newton second law (3) and equation (9), i.e. the equation of motion for particle of mass m in a frame of reference whose origin has linear acceleration a and an angular velocity $\vec{\Omega}$ with respect to the reference frame [2].

Since we want to find out an "electromagnetic analogy" for the inertial forces, then we can set F=0. The equation of motion (9) then can be derived from Lagrangian L=T-V, where T is the kinetic energy and V is a velocity-dependent generalized potential [2]:

$$V(x, \dot{x}, t) = ma \cdot x - m\dot{x} \cdot \vec{\Omega} \times x - \frac{m}{2}\left(\vec{\Omega} \times x\right)^2, \tag{10}$$

Which is a linear function of the velocities. We now may consider that the right hand side of equation (10) consists of a scalar potential [2]:

$$\phi(x,t) = ma \cdot x - \frac{m}{2}\left(\vec{\Omega} \times x\right)^2, \tag{11}$$

And a vector potential:

$$A(x,t) \equiv m\dot{x} \cdot \vec{\Omega} \times x, \tag{12}$$

So that

$$V(x,\dot{x},t) = \phi(x,t) - \dot{x} \cdot A(x,t). \tag{13}$$

Then the equation of motion (9) may now be written in Lorentz form as follows [2]:

$$m\ddot{x} = E(x,t) + x \times H(x,t), \tag{14}$$

With

$$E = -\frac{\partial A}{\partial t} - \nabla\phi = -m\Omega \times x - ma + m\Omega \times (x \times \Omega), \tag{15}$$

And

$$H = \nabla \times A = 2m\Omega. \tag{16}$$

At this point we may note [2, p. 303] that Maxwell equations are satisfied by virtue of equations (15) and (16). The correspondence between Coriolis force and magnetic force, is known from Larmor method. What is interesting to remark here, is that the same result can be expected directly from the basic equation of Quaternion Space (3) [1].

The aforementioned simplified approach indicates that it is indeed possible to find out Maxwell equations in Quaternion space, in particular based on our intuition of the direct link between Newton second law in Q-space and Feynman's derivation of Maxwell equations based on Lorentz force.

As an added note, we can mention here, that the aforementioned Feynman's derivation of Maxwell equations is based on commutator relation which has classical analogue in the form of Poisson bracket. Then there can be a plausible way to extend directly this 'classical' dynamics to quaternion extension of Poisson bracket [14], by assuming the dynamics as element of $r \in H \wedge H$ of the type: $r = ai \wedge j + bi \wedge k + cj \wedge k$, from which we can define Poisson bracket on H. But in the present paper we don't explore yet such a possibility.

In the next section we will discuss more detailed derivation of Maxwell equations in Q-space, by virtue of Gersten's method of Dirac decomposition [4].

4 A new derivation of Maxwell equations in Quaternion Space by virtue of Dirac decomposition.

In this section we present a derivation of Maxwell equations in Quaternion space based on Gersten's method in order to derive Maxwell equations from one photon equation by virtue of Dirac decomposition [4]. It can be noted here that there are other methods to derive such 'quantum Maxwell equations' (that is to find link between photon equation and Maxwell equations), for instance by Barut (see ICTP preprint no. IC/91/255).

We know that Dirac deduces his equation from the relativistic condition linking the Energy E, the mass m and the momentum p [5]:

$$(E^2 - c^2\vec{p}^2 - m^2c^4)I^{(4)}\Psi = 0, \tag{17}$$

Where $I^{(4)}$is the 4x4 unit matrix and $\Psi$is a 4-component column (bispinor) wavefunction. Dirac then decomposes equation (17) by assuming them as a quadratic equation:

$$(A^2 - B^2)\Psi = 0 \tag{18}$$

Where

$$A = E, \tag{19}$$

$$B = c\vec{p} + mc^2. \tag{20}$$

The decomposition of equation (18) is well known, i.e. (A+B)(A-B)=0, which is the basic proposition of Dirac, that is to decompose equation (18) into 2x2 unit matrix and Pauli matrix [4][12].

By virtue of the same method with Dirac, Gersten found in 1999 [4] a decomposition of one photon equation from relativistic energy condition (for massless photon [5]):

$$(\frac{E^2}{c^2} - \vec{p}^2)I^{(3)}\Psi = 0, \tag{21}$$

Where $I^{(3)}$ is the 3x3 unit matrix and $\Psi$is a 3-component column wavefunction. Gersten then found that equation (21) decomposes into the form [4]:

$$[\frac{E}{c}I^{(3)} - \vec{p}\cdot\vec{S}][\frac{E}{c}I^{(3)} + \vec{p}\cdot\vec{S}]\vec{\Psi} - \begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix}(\vec{p}\cdot\vec{\Psi}) = 0, \tag{22}$$

Where $\vec{S}$is a spin one vector matrix with components [4]:

$$S_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & -i & 0 \end{pmatrix}, \tag{23}$$

$$S_y = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \tag{24}$$

$$S_z = \begin{pmatrix} 0 & -i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{25}$$

And with the properties:

$$[S_x, S_y] = iS_z, \quad [S_x, S_z] = iS_y, [S_y, S_z] = iS_x, \vec{S}^2 = 2I^{(3)}. \tag{26}$$

Gersten asserts that equation (22) will be satisfied if the two equations [4][5]:

$$[\frac{E}{c} I^{(3)} + \vec{p} \cdot \vec{S}]\vec{\Psi} = 0, \tag{27}$$

$$\vec{p} \cdot \vec{\Psi} = 0, \tag{28}$$

are simultaneously satisfied. The Maxwell equations [9] will be obtained by substitution of E and p with the ordinary quantum operators (see for instance Bethe, *Field Theory*):

$$E \rightarrow i\hbar \frac{\partial}{\partial t} \quad , \tag{29}$$

and

$$p \rightarrow -ih\nabla, \tag{30}$$

And the wavefunction substitution:

$$\vec{\Psi} = \vec{E} - i\vec{B}, \tag{31}$$

Where E and B are electric and magnetic fields, respectively. With the identity:

$$(\vec{p} \cdot \vec{S})\vec{\Psi} = \hbar\nabla \times \vec{\Psi}, \tag{32}$$

Then from equation (27) and (28) one will obtain:

$$i\frac{\hbar}{c}\frac{\partial(\vec{E} - i\vec{B})}{\partial t} = -\hbar\nabla \times (\vec{E} - i\vec{B}), \tag{33}$$

$$\nabla \cdot (\vec{E} - i\vec{B}) = 0, \tag{34}$$

Which are the Maxwell equations if the electric and magnetic fields are real [4][5].

We can remark here that the combination of E and B as introduced in (31) is quite known in literature [6][7]. For instance, if we use positive signature in (31), then that is called as Bateman representation of Maxwell equations ($div\vec{\varepsilon} = 0; rot\vec{\varepsilon} = \frac{1}{c}\frac{\partial\vec{\varepsilon}}{\partial t}; \varepsilon = \vec{E} + i\vec{B}$). But the equation (31) with negative signature represents the *complex nature* of Electromagnetic fields [6], which suggests that these fields can also be represented in quaternion form.

Now if we represent in other form $\vec{\varepsilon} = \vec{E} - i\vec{B}$ as more conventional notation, then equation (33) and (34) can be written in simple form, as follows:

$$i\frac{\hbar}{c}\frac{\partial\vec{\varepsilon}}{\partial t} = -\hbar\nabla \times \vec{\varepsilon}, \tag{35}$$

$$\nabla \cdot \vec{\varepsilon} = 0. \tag{36}$$

Now to consider quaternionic expression of the above results from Gersten [4], one can begin with the same linearization procedure just as described in equation (5):

$$dz = (dx_k + idt_k)\, q_k,$$

Which can be viewed as the quaternionic square root of the metric interval dz:

$$dz^2 = dx^2 - dt^2. \tag{37}$$

Now consider the relativistic energy condition (for massless photon [5]) similar to equation (21):

$$E^2 = p^2c^2 \Rightarrow (\frac{E^2}{c^2} - \vec{p}^2) = k^2, \tag{38}$$

It is obvious that equation (38) has the same form of quadratic equation with (37), therefore we can find its quaternionic square root too, as follows:

$$k = \left(E_{qk}/c^2 + i\vec{p}_{qk}\right) q_k, \tag{39}$$

Where q represents the quaternion unit matrix, then equation (5) actually is a matrix-valued metric but we do not discuss that further in the present paper. Therefore the linearized quaternion root decomposition of equation (21) can be written as follows [4]:

$$\begin{array}{l} [\frac{E_{qk}q_k}{c}I^{(3)} + i\vec{p}_{qk}q_k \cdot \vec{S}][\frac{E_{qk}q_k}{c}I^{(3)} + i\vec{p}_{qk}q_k \cdot \vec{S}]\vec{\Psi} \\ - \begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix} (i\vec{p}_{qk}q_k \cdot \vec{\Psi}) = 0 \end{array} \quad . \quad . \tag{40}$$

In accordance with Gersten's method [4], and extending his method to quaternion situation, then equation (40) will be satisfied if the two equations:

$$[\frac{E_{qk}q_k}{c}I^{(3)} + i\vec{p}_{qk}q_k \cdot \vec{S}]\vec{\Psi}_k = 0, \tag{41}$$

$$i\vec{p}_{qk}q_k \cdot \vec{\Psi}_k = 0, \tag{42}$$

are simultaneously satisfied. Now we introduce similar wavefunction substitution, but this time in quaternion form:

$$\vec{\Psi}_{qk} = \vec{E}_{qk} - i\vec{B}_{qk} = \vec{\varepsilon}_{qk}. \tag{43}$$

And with the identity:

$$(\vec{p}_{qk}q_k \cdot \vec{S})\vec{\Psi}_k = \hbar\nabla_k \times \vec{\Psi}_k \quad . \tag{44}$$

Then from equation (41) and (42) one will obtain *the Maxwell equations in Quaternion-space* as follows:

$$i\frac{\hbar}{c}\frac{\partial\vec{\varepsilon}_{qk}}{\partial t} = -\hbar\nabla_k \times \vec{\varepsilon}_{qk} \quad , \tag{45}$$

$$\nabla_k \cdot \vec{\varepsilon}_{qk} = 0. \tag{46}$$

Now the remaining question is how to define quaternion differential operator in the right hand side of (45) and left hand side of (46).

In order to define the complete description of the left-hand side of equation (46), one can choose a definition of quaternion differential operator, that is the Moisil-Theodoresco operator[8] as follows:

$$D[\varphi] = grad\varphi = \sum_{k=1}^{3} i_k \partial_k \varphi = i_1 \partial_1 \varphi + i_2 \partial_2 \varphi + i_3 \partial_3 \varphi, \tag{47}$$

Where we can define here that $i_1 = i, i_2 = j, i_3 = k$ to represent 2x2 quaternion unit matrix, for instance.

Therefore the first derivative of equation (43) now can be expressed in similar notation of (47) :

$$\begin{array}{l} D[\vec{\Psi}] = D[\vec{\varepsilon}] = i_1 \partial_1 E_1 + i_2 \partial_2 E_2 + i_3 \partial_3 E_3 \\ -i\left(i_1 \partial_1 B_1 + i_2 \partial_2 B_2 + i_3 \partial_3 B_3\right) \end{array} . \tag{48}$$

This expression indicates that both electric and magnetic fields can be represented in unified manner in a biquaternion form.

Then we define quaternion differential operator in the right-hand-side of equation (45) by a quaternion extension of the curl operator, as described as follows:

$$\nabla \times A_{qk} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}, \tag{49}$$

where i,j,k represents quaternion matrix as described above. This quaternionic extension of curl operator is based on the known relation of multiplication of two arbitrary complex quaternions q and b as follows:

$$q \cdot b = q_0 b_0 - \left\langle \vec{q}, \vec{b} \right\rangle + [\vec{q} \times \vec{b}] + q_0 \vec{b} + b_0 \vec{q}, \tag{50}$$

Where

$$\left\langle \vec{q}, \vec{b} \right\rangle := \sum_{k=1}^{3} q_k b_k \in C, \tag{51}$$

And

$$\left[\vec{q} \times \vec{b}\right] := \begin{vmatrix} i & j & k \\ q_1 & q_2 & q_3 \\ b_1 & b_2 & b_3 \end{vmatrix}. \tag{52}$$

We can note here that there could be more rigorous approach to define such a quaternionic curl operator.[7]

In the present paper we only discuss derivation of Maxwell equations in Quaternion Space using the decomposition method as described by Gersten [4][5]. Further extension of this proposition in order to derive Proca equations in Quaternion Space appears possible too using the same method [5], but we will not discuss this extension here.

In the next section we will discuss some physical implications of this new derivation of Maxwell equations in Quaternion Space.

5 A few implications: de Broglie's wavelength and spin

In the foregoing section we derived a consistent description of Maxwell equations in Q-Space by virtue of Dirac-Gersten's decomposition. Now we discuss some plausible implications of the new proposition.

First, in accordance with Gersten, we submit the viewpoint that the Maxwell equations yield wavefunctions which can be used as guideline for interpretation of quantum mechanics [4][5]. The one-to-one correspondence between classical and quantum wave actually can be expected not only in the context of Feynman's derivation of Maxwell equations from Lorentz force, but also from known exact correspondence between commutation relation and Poisson bracket [2][3]. Furthermore, the proposed quaternion form of Maxwell equations in Q-space yields to a novel viewpoint of both the wavelength, as discussed below, and also a plausible mechanical model of spin [13].

The equation (39) implies that momentum and energy could be expressed in quaternion form. Now by introducing the definition of de Broglie's wavelength ($\lambda_{DB} = \frac{\hbar}{p} \rightarrow p_{DB} = \frac{\hbar}{\lambda}$), then one obtains an expression of equation (39) in terms of wavelength:

$$k = \left(E_k/c^2 + i\vec{p}_k\right) q_k = \left(E_k q_k/c^2 + i\vec{p}_k q_k\right) = \left(E_k q_k/c^2 + i\frac{\hbar}{\lambda_k^{DB} q_k}\right) \quad (53)$$

In other words, now we can express de Broglie's wavelength in a consistent Q-basis:

$$\lambda_{DB-Q} = \frac{\hbar}{\sum_{k=1}^{3}(p_k)q_k} = \frac{\hbar}{\sum}{}^{3}_{k=1}(v_{group}m_k)q_k, \quad (54)$$

Therefore the above equation can be viewed as an Extended De Broglie wavelength in Q-space. This equation means that the mass also can be expressed in Q-basis. In the meantime, a quite similar method to define quaternion mass has also been considered elsewhere [13], but he has not yet expressed his result in Dirac equation form as presented here.

The new result of the present paper is that we are able to derive Maxwell equations in Quaternion space based on Dirac decomposition, by virtue of Gersten's method. This proposition has never been submitted before, as far as our concern.

Further implications of this new proposition of quaternion de Broglie wavelength and also quantum mechanical model of spin require further study, and therefore these issues are not discussed from the present paper.

6 Concluding remarks

In the present paper we derive a consistent description of Maxwell equations in Quaternion space. First we consider a simplified method similar to the Feynman's derivation of Maxwell equations from Lorentz force. And then we present another method to derive Maxwell equations by virtue of Dirac decomposition, as described by Gersten (1999).

In accordance with Gersten, we submit the viewpoint that the Maxwell equations yield wavefunctions which can be used as guideline for interpretation of quantum mechanics. The one-to-one correspondence between classical wave and quantum wave as described here actually can be expected not only in the context of Feynman's derivation of Maxwell equations from Lorentz force, but also from the known exact correspondence between commutation relation and Poisson bracket [2][4].

A new implication obtained from the above results of Maxwell equations in Quaternion Space, is that it suggests that the DeBroglie wavelength will have quaternionic form. Its further implications, however, are beyond the scope of the present paper.

In the present paper we only discuss derivation of Maxwell equations in Quaternion Space using the decomposition method as described by Gersten [4][5]. Further extension to Proca equations in Quaternion Space appears possible too using the same method as described here.

The proposition of Maxwell equations in Quaternion space as described here deserves further theoretical considerations. Further observation of some implications of this result is recommended.

Acknowledgement

This writer wishes to express his gratitude to Jesus Christ, who have saved him. He is the Good Shepherd (Psalm 23).

The writer of this article is grateful for kindness and hospitality extended to him by Professor A.P. Yefremov and Professor M. Fil'chenkov in the People's Friendship University of Russia (PFUR), at Institute of Gravitation and Cosmology. The writer of this article would like to express thanks to Professor V.V. Kassandrov for excellent guide to Maxwell equations, and to Professor Y.P. Rybakov for discussion on possibility of quaternion de Broglie's wavelength. The writer of this article is also grateful for lectures by Professor Mikhail Fil'chenkov, Professor Vladimir D. Ivashchuk, Professor Kyril Bronnikov, Professor Yu Vladimirov. The writer also express his thanks for kindness and encouraging support of Professor Florentin Smarandache and Dmitri Rabounski. The writer of this article, that is Victor Christianto, already repent and receive Jesus Christ. Jesus Christ will come again very very very soon, be hurry to repent and receive Jesus Christ.

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