Zeno's Achilles\Tortoise Race and Reconsiderations of Some Mathematical Paradigms

Doron Shadmi *shadmi_doron@yahoo.com*

Keywords: Organism, Non-locality, Locality, Distinction, Uncertainty, Redundancy, Researcher, Researched, Organic Number, "The Tree of Knowledge"

0 Abstract

An original observation of Zeno's Achilles\Tortoise Race Paradox is introduced. It leads to novel understanding of the foundations of mathematical science, especially by observing Non-locality and Locality as its fundamental building-blocks. Locality is precisely its own formula, thus this formula cannot be used as a solution for anything else but its own unique case. Non-locality is a formula that can be used as a solution for more than one case. Locality on its own is total isolation. Non-locality on its own is total connectivity. No total realm is researchable. A researchable realm only exists if Non-locality and Locality are not total. Under Non-locality\Locality Linkage we get a universe where Non-locality is its common law; this is expressed by many Localities that are gathered by the common law, but can never be Non-local, as is the common law. Non-locality\Locality Linkage can be perceived as "The Tree of Knowledge", which is the one organic and ever complex (and therefore non-entropic) realm that enables one, and only one simple law (Non-locality), to be the common knowledge of many Local expressions of it (we show that Leibniz Chaitin Complexity [11] Challenge is the organic incompleteness of Non-locality\Locality Linkage).

1 Zeno's Achilles\Tortoise Race

It is argued that Zeno's Achilles\Tortoise Race is not a paradox in real life because we can summarize non-finite values (where each value > 0) that are added to some initial value. By doing that we are able to get an accurate value, which is different from the initial value. For example: 1 is the initial value and $1+1/2+1/4+1/8+\ldots= 2$, where 2 is an accurate value that is different from the initial value 1. Actually the whole idea of Limits is somehow motivated by the desire to solve Zeno's Paradox. Let us investigate three different cases of Achilles\Tortoise Race:

Case A: Achilles wins against the Tortoise, and the Race stops.

Case B: Achilles does not win against the Tortoise, and the Race continues (actually forever).

Case C: Achilles and the Tortoise are on the same position and the Race stops.

Case A:

Distance = Speed * Time

The next position of Achilles and the Tortoise along the Race = previous position + Distance.

Case A exists only if neither Speed nor Time are changed during the Race. Let us show it by using an algorithm (no particular programming language is used here):

```
Position X1 = 0

Position X2 = 10

Achilles Speed = Aspeed = 10

Tortoise Speed = Tspeed = 1

Time = 1

Do Loop K from 1 to ∞

    Achilles position = position X1 + distance ( = Aspeed * Time)

    Tortoise position = position X2 + distance (= Tspeed * Time)

    Position X1 = Achilles position

    Position X2 = Tortoise position

    If X1 > X2 then STOP

Next Loop K

Loop K 1:

  Achilles position = 0 + (10 * 1) = 10

  Tortoise position = 10 + (1 * 1) = 11

  Position X1 = Achilles position = 10

  Position X2 = Tortoise position = 11
```

(The Race continues after `Loop K 1`, since Tortoise position > Achilles position)

```
Loop K 2:

  Achilles position = 10 + (10 * 1) = 20

  Tortoise position = 11 + (1 * 1) = 12

  Position X1 = Achilles position = 20

  Position X2 = Tortoise position = 12
```

(The Race stops after `Loop K 2`, since Tortoise position < Achilles position)

Distance = Speed * Time

In *case B,* Time is changed during the Race. Let us show it by using an algorithm (no particular programming language is used here):

```
Position X1 = 0
Position X2 = 10
Achilles Speed = Aspeed = 10
Tortoise Speed = Tspeed = 1
Time = 1

Do Loop K from 1 to ∞

   Achilles position = position X1 + distance ( = Aspeed * Time)

   Tortoise position = position X2 + distance (= Tspeed * Time)

   Position X1 = Achilles position

   Position X2 = Tortoise position

   If X1 ≥ X2 then STOP

   Time = Time / Aspeed (Achilles Speed = Aspeed = 10)
Next Loop K


Loop K 1:

  Achilles position = 0 + (10 * 1) = 10

  Tortoise position = 10 + (1 * 1) = 11

  Position X1 = Achilles position = 10

  Position X2 = Tortoise position = 11

  Time = Time / Aspeed (Achilles Speed = Aspeed = 10) = 0.1
```

  (The Race continues after Loop K 1)

```
Loop K 2:

  Achilles position = 10 + (10 * 0.1) = 11

  Tortoise position = 11 + (1 * 0.1) = 11.1

  Position X1 = Achilles position = 11

  Position X2 = Tortoise position = 11.1

  Time = Time / Aspeed (Achilles Speed = Aspeed = 10) = 0.01
```

  (The Race continues after Loop K 2)

```
Loop K 3:

  Achilles position = 11 + (10 * 0.01) = 11.1

  Tortoise position = 11.1 + (1 * 0.01) = 11.11

  Position X1 = Achilles position = 11.1

  Position X2 = Tortoise position = 11.11

  Time = Time / Aspeed (Achilles Speed = Aspeed = 10) = 0.001
```

(The Race continues after `Loop K 3`)

```
...

Loop K ∞
```

The Race continues forever because any given position of Achilles and the Tortoise that comes next is the result of previous positions + Distances values, where each Distance value > 0, no matter how many loops (finite or infinite) are used (Achilles position < Tortoise position is an invariant state).

Claim 1: Since Achilles position < Tortoise position is an invariant state of *Case B*, and since *Case C* is exactly the *Case B* algorithm, we conclude that *Case C* does not apply.

Claim 2: One claims that (Time > 0) / Achilles Speed is *Case B*, but *Case C* is exactly `Loop K ∞`, where at `Loop K ∞` we have (Time = 0) / Achilles Speed.

Let us carefully investigate `Loop K ∞` according to *Claim 2:*

```
Loop K 3:

  Achilles position = 11 + (10 * 0.01) = 11.1

  Tortoise position = 11.1 + (1 * 0.01) = 11.11

  Position X1 = Achilles position = 11.1

  Position X2 = Tortoise position = 11.11

  Time = Time / Aspeed (Achilles Speed = Aspeed = 10) = 0.001
```

(The Race continues after `Loop K 3`)

```
...
Last Apos =
```
The last `Achilles position` for some `K < ∞`

```
Last Tpos =
```
The last `Tortoise position` for some `K < ∞`

```
Ltime =
```
The last `Time` unit's change for some `K < ∞`

`Last Apos` < `Last Tpos` as long as `K < ∞`
```
...
```

```
Loop K ∞:

  Achilles position = Last Apos + (Aspeed * Ltime) = Current Apos

  Tortoise position = Last Tpos + (Tspeed * Ltime) = Current Tpos

  Position X1 = Achilles position = Current Apos

  Position X2 = Tortoise position = Current Tpos

  Ltime = 0 / Aspeed (Achilles Speed = Aspeed = 10)
```

The Race continues at `Loop K ∞` state, since `Current Apos < Current Tpos` is invariant exactly because `Ltime = 0/Aspeed` at `Loop K ∞`. So *Claim 2* and *Claim 1* are reducible to *Case B*.

It is also argued that at `Loop K ∞` we cannot use anything that applies at `Loop K < ∞`. Therefore *Claim 2* is not reducible to *Case B*. If this is the case then `Loop K ∞` consists of at least two extremes that are not derived from each other as follows: 1) The `Loop K ∞` realm is totally Local, but then `Achilles` or the `Tortoise` do not exist as two competitors and also there is no room for any Race. 2) The `Loop K ∞` realm is totally Non-local, but then `Achilles` or the `Tortoise` do not exist, since totally Non-local realm does not enable the existence of localities like `Achilles` or the `Tortoise`. So also at that extreme ∞ realm there are no competitors and therefore no Race. The only alternative for Achilles\Tortoise Race is a realm that results from Non-locality\Locality Linkage, which is not totally Non-local AND not totally Local. This linkage is exactly *Case B*. By following *Case B* `Current Apos < Current Tpos` invariant result it is argued that infinitely many positions are not the higher dimension that they cover. Actually "<" of an expression like "`Current Apos < Current Tpos`" is an example of a higher dimension that is "free" of any positions of lower dimensions on it, no matter how many positions of lower dimensions there may be between any arbitrary pair of lower dimensions' positions. Non-locality\Locality Linkage actually solves the philosophical problem of Motion, because the researched framework is not understood only in terms of Locality.

3 Fundamental Physics

At the beginning of the 20th century there were two major revolutions in Physics, when STR [1], GRT [2] and QM first aired their views. These theories provided better explanations for physical anomalies both in macro and the micro scales. At the beginning of the 21st century there is still no simple and straightforward theory that naturally links the macro and the micro scales. Nevertheless both macro and micro frameworks have changed their attitudes about the researcher as a significant factor of the researched framework. Nowadays the researcher is no longer considered as a "pure" observer of the results, and the researcher's possible influence on the results is not unconditionally ignored. Also properties like Uncertainty, Redundancy and Randomness etc … are increasingly understood to be essentials of the researched space.

In 1935 the 'EPR thought experiment' was published [3], and since then it has led an approach that disagrees with the probabilistic interpretations of physical theories and experimental results. EPR's original aim was to show that QM's probabilistic interpretations enable non-local phenomena that allow information to be transferred faster than the speed of light (which is an essential constant of Einstein's SRT and GRT). Actual physical experiments [4], mostly

based on Bell [5] and Bohm [6] thought experiments, have shown non-local phenomena, but until now there is no agreement about the Bell and Bohm thought experiments, and there is no agreement about the experimental results that are based on them, including disagreements about their technical validity and\or their interpretations. It has to be stressed that at the base of both experimental and theoretical frameworks there is the mathematical language, and some scholars believe that a better understanding of that language can display these aforementioned problems in a new light. These voices can be added to the vision of L. Lovasaz "One Mathematics" [7], the aim of which is to find a common base ground for many branches of mathematics under one organic approach [8], which enables better linkage among them.

Here are some suggested abstract principles that may express these notions:

1. Atomic state is an existing thing that has no sub-existing things, for example: {} is the Atomic state, where {{}} is an example of a non-atomic state.

2. In order to be expressed beyond the Atomic state, we suggest at least two existing things that are linked with each other, without deriving from each other.

3. We observe Non-locality and Locality as two existing things that, if linked with each other, enable the expression of things beyond the Atomic state. In that case, Distance, Division, Scale, Size or any other possible terms of measurement are based on Non-locality\Locality Linkage.

Non-locality\Locality Linkage can be introduced, for example, by the relation between different dimensions. A. Khrennikov uses the analogy that Non-Euclidean local effects might also be imagined as non-local Euclidean effects [9]. This analogy is transferable to the analogy of different dimensions, as follows:

$n=1$ to ∞
$k= 0$ to $n-1$

What is considered as non-local in the sense of simultaneous correlation between k-dim elements, is the non-local property of n-dim element with respect to these k-dim elements. In that case, any measurement among at least two dimensions is actually based on Non-locality\Locality Linkage (n-dim\k-dims linkage). By following this notion any n-dim is non-local with respect to any amount of k-dim elements because: given n-dim element, there are infinitely many k-dim elements on it, such as k-dimA $\neq$ k-dimB $\neq$ k-dimC …, where $\neq$ is an example of a n-dim domain, which is not covered by any k-dim element. If one claims against this assertion then he has to avoid $\neq$. But then there is, at most, one and only one k-dim element on the n-dim element. By carefully investigating the Dimensions' example it is discovered that $\neq$ is equivalent to n-dim which is used as a differentiator between k-dims, no matter what identification each k-dim has. In that case we distinguish between the state of many localities and the state of Non-locality (n-dim state > many k-dims state) where the identification of each k-dim has no significance. For example: k-dim $\neq$ k-dim $\neq$ k-dim has cardinality 3, whether the k-dims are distinguishable or not. By extending the cardinality to ∞, we still find $\neq$ as a projection of n-dim, which is used as a differentiator between more than one k-dim on it. We think that these examples can be used Without Loss of Generality in many branches of Mathematics.

We observe the difference between Non-locality and Locality under their linkage as follows:

Locality under linkage is precisely its own formula, thus this formula cannot be used as a solution for anything else but its own unique case. Non-locality under linkage is exactly a formula that can be used as a solution for more than one single case. Under Non-locality\Locality Linkage we get a universe where Non-locality is its common law, which is expressed by many Localities that are gathered by common law, but can never be Non-local, as common law is. Under Non-locality\Locality Linkage we have a naturally non-entropic realm, which is derived from the principle of non-finite progression that is the result of the openness of the Non-Local and Local, with respect to each other. This generalization may provide a non-trivial basis for what we call "The Organic Unity of Mathematical Science" [10].

Organism means that any possible expression exists simultaneously together with the entirety of expressions, so that each expression enables the non-trivial further development of the remaining expressions. It means that we have a non-trivial framework that is based on invariant principles, which are actually expressed by many different and non-trivial ways. In order to lift things up (to develop them) we simultaneously need stability AND changeability in one framework (*"Give me a place to stand on, and I will move the Earth"*, **Archimedes of Syracuse**). Let us use Non-locality\Locality Linkage in order to investigate a concept like Distinction. First let us provide some definitions:

x is an element.

Definition 1: **Identity** is a property of *x*, which allows its recognition.
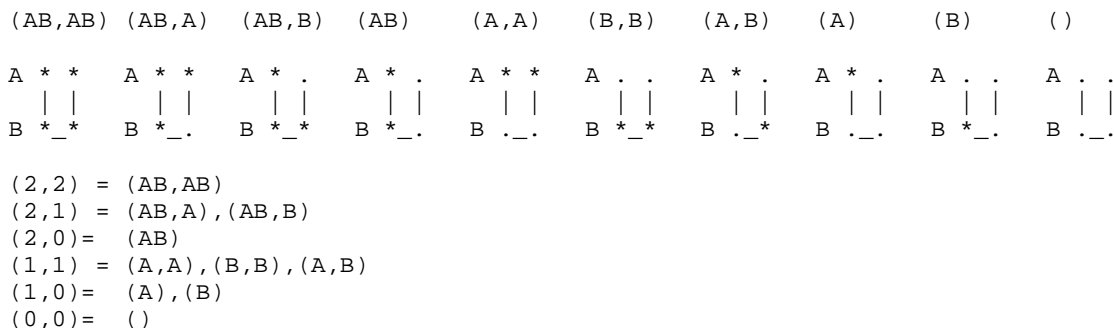
Definition 2: **Copy** is a duplication of a single **identity**.

Definition 3: If *x* has more than one single **identity**, then *x* is called **Uncertain**.

Definition 4: If *x* has more than one single **copy**, then *x* is called **Redundant**.

The linkage between Non-locality and Locality defines a Distinction-Tree, where its Y-axis is used in order to measure the Uncertainty of its elements, and its X-axis is used in order to measure the Redundancy of its elements.

For example, the 2-Uncertainty ✗ 2-Redundancy Distinction-Tree is:

```
(AB,AB) (AB,A)  (AB,B)  (AB)     (A,A)    (B,B)    (A,B)    (A)      (B)      ()

A * *   A * *   A * .   A * .   A * *   A . .   A * .   A * .   A . .   A . .
  | |     | |     | |     | |     | |     | |     | |     | |     | |     | |
B *_*   B *_.   B *_*   B *_.   B ._.   B *_*   B ._*   B ._.   B *_.   B ._.

(2,2) = (AB,AB)
(2,1) = (AB,A),(AB,B)
(2,0)=  (AB)
(1,1) = (A,A),(B,B),(A,B)
(1,0)=  (A),(B)
(0,0)=  ()
```

Any appearance of that tree is called Distinction State (DS), where any DS is under a structure called Frame (F), for example: (AB,B) is a DS that is under (2,1) F. The order in each DS or F has no significance (similar to {a,b}={b,a}) but any DS is the basis of any possible order (similar to the concept of Set as being the basis of permutations).

Organic Numbers (ON) is a number system that is based on DS.

We do not need more than A and B in order to introduce the entire ON system.
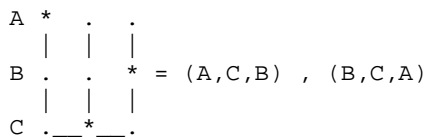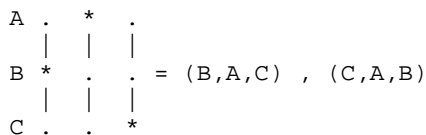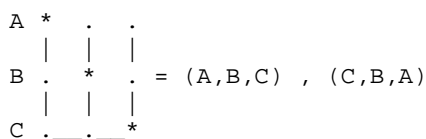
1) AB represents the Uncertain state of Distinction of an element under ON.

2) A,B represents the clear identification state of Distinction under ON.

3) A,A represents the Redundant state of Distinction under ON.

(1)(2)(3) itself can be represented as ABC ; A,B,C ; A,A,A cases of ON3.

The principle here is to reduce things into a single structure (a one organism). For example: we need 3 2-dim structures in order to represent the 6 permutations of (A,B,C) structure:

```
A  *   .   .
   |   |   |
B  .   *   .   =  (A,B,C) , (C,B,A)
   |   |   |
C  .__.__*


A  .   *   .
   |   |   |
B  *   .   .   =  (B,A,C) , (C,A,B)
   |   |   |
C  .__.__*


A  *   .   .
   |   |   |
B  .   .   *   =  (A,C,B) , (B,C,A)
   |   |   |
C  .__*__.
```

But (A,B,C) case is reducible into 3-dim 3-Uncertainty x 3-Redundancy organism:
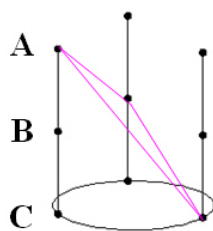
So the A,B,C identifications can be picked in any desired order, under the same ON.

4 The metaphysical view of Zeno's Achilles\Tortoise Race

Zeno's Achilles\Tortoise Race actually deals metaphysically with the one-many relation as appears between the Eleatic and Pluralists schools of ancient Greece [12]:

In fact all mathematical attempts to resolve these paradoxes share a common feature, a feature that makes them consistently miss the fundamental point which is Zeno's concern for the one-many relation, or it would be better to say, lack of relation. This takes us back to the ancient dispute between the Eleatic school and the Pluralists. The first, following Parmenide's teaching, claimed that only the One or identical can be thought and is therefore real, the second held that the Many of becoming is rational and real.

The one-many relation actually stands at the base of any solution of this paradox, whether it is continuous or discrete. In both cases two competitors and a one race path must exist, so the minimal conditions of Zeno's Achilles\Tortoise Race must be based on one-many relation. A luck of such a relation actually prevents the race's existence, as already written:

1) The Loop K ∞ realm is totally Local, but then Achilles or the Tortoise do not exist as two competitors and also there is no room for any Race. 2) The Loop K ∞ realm is totally Non-local, but then Achilles or the Tortoise do not exist, since totally Non-local realm does not enable the existence of localities like Achilles or the Tortoise. So also at that extreme ∞ realm there are no competitors and therefore no Race. The only alternative for Achilles\Tortoise Race is a realm that results from Non-locality\Locality Linkage, which is not totally Non-local AND not totally Local.

In other words, a one-many relation must exist, where both schools are in relations, and these relations are demonstrated as follows:

a) By distinguish between the Local and The Non-local we show how both cases ("*Case A*: Achilles wins against the Tortoise, and the Race stops" or "*Case B*: Achilles does not win against the Tortoise, and the Race continues (actually forever)") are both valid, (depending on our interruption of the parameters of the formula *Distance = Speed * Time* during the race) (According to this view *Case C* is already included in *Case A*).

b) The solution is compared with the current mathematical paradigm that solves this paradox by claiming that infinitely many values (where each one of them > 0) can be summarized to some accurate value, which is different than the initial value. By carefully analyzing **Case *B*** and by using the difference between Non-locality and Locality, we explicitly show that the currant mathematical paradigm that solves *Zeno's Achilles\Tortoise Race* paradox is wrong.

c) In order to understand why the current mathematical paradigm is wrong all we have is to distinguish between the minimal form of Non-locality, which is a single endless line (where only 1-dimesion is considered) and the minimal form of Locality, which is a single point (where only 0-dimension is considered).

d) Exactly as no line segment (which is the result of the linkage between Non-locality AND Locality) can be an endless line, then no line segment can be a point. By using this symmetrical inability, we immediately understand that the currant mathematical solution of *Zeno's Achilles\Tortoise Race* paradox is wrong.

e) The new paradigm, which is based on the qualitative difference between Non-locality and Locality, fundamentally changes Standard and Non-standard Real Analysis.

f) Also our understanding of the concept of Collection is fundamentally changed, because now we realize that no collection is totally non-local and no collection is totally local (as a result the term "all" is invalid if it is related to an infinite collection because no infinite extrapolation of elements is Non-locality, and no infinite interpolation of elements is Locality).

g) By following (**f**) no element of an infinite collection is considered as its final element, and therefore any infinite collection is incomplete. On the contrary, the term "all" is valid only in the case of a finite collection, where any element is considered as its final element and therefore any finite collection is complete (the extension from the finite to the infinite used by Cantor and other mathematicians or philosophers, is wrong, at the moment that we understand the qualitative difference between Non-locality and Locality).

h) As for Non-locality and Locality, they are two opposite aspects of a one atomic state, as follows:

An atom is an existing thing that has no sub-exiting things, for example:

The empty set (notated as {}) is an existing thing that has no sub-exiting things.

The minimal version of the non-local aspect of the atomic state is exactly an open endless line (where only 1-dim is considered).

The minimal version of the local aspect of the atomic state is exactly a point (where only 0-dim is considered).

i) Non-locality\Locality linkage is the exact condition that enables Complexity, whether it is abstraction or not. One of the possible phenomena is Zeno's Achilles\Tortoise Race, where both *Case A* and **Case B** are valid.

5 Summary

ZFC set theory (which is used as the foundation for many branches of Mathematics) uses only A,B as the general case of Distinction, while ON are based on a generalization of Distinction (as shown in pages 7 and 8). Furthermore, since ON are the results of Non-locality\Locality linkage, where this linkage is one "Tree of Knowledge", then the researcher and the researched are organs of one organism. Currently the organic approach is not at the core of mathematical science. On the contrary, any discovered\invented linkage between some given branches of Mathematics is considered as an unexpected event of this science, and we argue that this arbitrary paradigm has to be developed into the organic view of mathematical science.

We believe that this is the right way to unify the abstract and the non-abstract realms into one fruitful and dynamically-developed body of scientific knowledge, where researcher\researched linkage is not unconditionally ignored. Non-locality\Locality Linkage can be perceived as "The Tree of Knowledge", which is the one organic and ever-complex (and therefore non-entropic) realm that enables one and only one simple law (Non-locality) to be the common knowledge of many Local expressions of it (we have shown that Leibniz Chaitin Complexity [11] Challenge is the organic incompleteness of Non-locality\Locality Linkage).

Acknowledgements

References

[1] Einstein Albert: *On the Electrodynamics of Moving Bodies*, Annalen der Physik, 17:891, June 30, 1905 (English translation by W. Perrett and G.B. Jeffery).

[2] Einstein Albert (1915): "Die Feldgleichungen der Gravitation (The Field Equations of Gravitation)", *Koniglich Preussische Akademie der Wissenschaften*: 844–847.

[3] Einstein, Podolsky, Rosen**:** *Can Quantum-Mechanical Description of Physical Reality Be Considered Complete?* Phys. Rev. 47, 777 - 780 (1935).

[4] Aspect Alain**:** *Experimental Tests of Realistic Local Theories via Bell's Theorem*, Phys. Rev. Lett. 47, 460 (1981).

[5] J. S. Bell, *On the Einstein Podolsky Rosen Paradox*, Physics 1, 195 (1964).

[6] Bohm David (1952). "*A Suggested Interpretation of the Quantum Theory in Terms of "Hidden Variables" I"*. Physical Review 85: 166–179

[7] L. Lovasz: *One Mathematics* http://www.cs.elte.hu/~lovasz/berlin.pdf .

[8] Moshe Klein, Doron Shadmi: *Organic Mathematics*, International Journal of Pure and Applied Mathematics, volume 49 No. 3 2008, 329-340 http://www.geocities.com/complementarytheory/IJPAM-OM.pdf .

[9] A. Khrennikov, *Interpretations of Probability*, Publisher: Walter de Gruyter; 2 edition (January 15, 2009ISBN-10: 3110207486ISBN-13: 978-3110207484)

[10] Hilbert David: *Mathematical Problems,* Bulletin of The American Mathematical Society, Volume 37. Number 4, Pages 407-436, S 0273-0979(00)00881-8.

[11] Cristian S. Calude *Randomness and Complexity, From Leibniz to Chaitin*, Publisher: World Scientific Publishing Company, date: Oct 2007, ISBN: 978-981-277-082-0 978-981-277-082-8

[12] Alba Papa-Ggimaldi *The Review of Metaphysics* 50 (December 1996): 299-314.