Pythagoras, music, sacred geometry, and genetic code

M. Pitkänen
Email: matpitka@luukku.com.
http://tgdtheory.com/public_html/.

November 8, 2014

Contents

Abstract

The idea that the 12-note scale could allow mapping to a closed path covering all vertices of icosahedron having 12 vertices and not intersecting itself is attractive. Also the idea that the triangles defining the faces of the icosahedron could have interpretation as 3-chords defining the notion of harmony for a given chord deserves study. The paths in question are known as Hamiltonian cycles and there are 1024 of them. There paths can be classified topologically by the numbers of triangles containing 0, 1, or 2 edges belonging to the cycle representing the scale. Each topology corresponds to particular notion of harmony and there are several topological equivalence classes.

I have also played with the idea that the 20 amino-acids could somehow correspond to the 20 triangles of icosahedron. The combination of this idea with the idea of mapping 12-tone scale to a

Hamiltonian cycle at icosahedron leads to the question whether amino-acids could be assigned with a topological equivalence class of Hamiltonian cycle and whether topological characteristics could correspond to physical properties of amino-acids. It turns out that the identification of 3 basic polar amino-acids with triangles containing no edges of the scale path, 7 polar and acidic polar amino-acids with those containing 2 edges of the scale path, and 10 non-polar amino-acids with triangles containing 1 edge on the scale path would be consistent with the constraints on the Hamiltonian cycles. One could of course criticize the lumping of acidic polar and polar aminoacids to same group.

The number of DNAs coding for a given amino-acid could be also seen as such a physical property. The model for dark nucleons leads to the vertebrate genetic code with correct numbers of DNAs coding for amino-acids. It is however far from clear how to interpreted DNAs geometrically and the problem whether one cold understand genetic code geometrically remains open.

1 Introduction

The conscious experiences generated by music demonstrate a fascinating connection between algebra and emotions. How can major and minor scale using different frequency ratios generate so different emotional experiences. This strongly suggests the we experience music as entire time interval, 4-D patterns - rather than time=constant snapshots. Also the ability remember the key and the tension lasting as long as the return to the basic key has not taken place, is example of this. One of the key questions is why octaves - that is powers of 2 of the basic note of the scale - are experienced as equivalent? One can also wonder what is behind consonance and dissonance.

I have already earlier tried to understand music experience and considered some ideas inspired by p-adic numbers fields - such as the idea that Pythagorean scale coming as powers of 3 for the basic note modulo octave equivalence might relate to 3-adicity. Reading of a book titled “Interference: A Grand Scientific Musical Theory” by Richard Merrick [J1] freely available in web (`http://interferencetheory.com/files/Interference.pdf`) re-stimulated my interest. In particular, I found the idea about a connection between music scale and harmonies with Platonic solids (3-D “sacred geometry”) as highly inspiring. The basic question was whether the 12-tone scale could be mapped to a curve going once through each point of icosahedron having 12 vertices and whether the 20 faces of icosahedron, which are triangles could define the basic chords in 12-tone scale. These curves are known as Hamiltonian cycles and in the case of icosahedron there are 2^{10} of them: those obtained from each other by rotation leaving icosahedron invariant are however equivalent.

A given triangle of icosahedron can contain 0, 1 or 2 edges of the cycle and the numbers of the triangles corresponding to these triangle types classify partially the notion of harmony characterized by the cycle. Quint cycle suggests the identification for the single edge of curve as quint interval so that triangles would represent basic 3-chords of the harmony with 0,1, or 2 quints.

One can make same questions also for other Platonic solids- tetrahedron (4 vertices), octahedron and cube which are duals of each other and have (6 and 8 vertices respectively, and dodecahedron which is dual of icosahedron having 20 vertices and 12 faces. Arabic music uses half intervals and scales with 19 and 24 notes are used. Could 20-note scale with harmony defined by 5-chords assigned to the pentagons of dodecahedron have some aesthetic appeal? Nowadays it is possible to develop electronically music based on this kind of scale and this kind of experimentation might be a fascinating intellectual and artistic adventure for a young composer.

I have also played with the idea that the 20 amino-acids could somehow correspond to the 20 triangles of icosahedron. The combination of this idea with the idea of mapping 12-tone scale to a Hamiltonian cycle at icosahedron leads to the question whether amino-acids could be assigned with the equivalence class of Hamiltonian cycles under icosahedral group and whether the geometric shape of cycle could correspond to physical properties of amino-acids [I1]. The identification of 3 basic polar amino-acids with triangles containing no edges of the scale path, 7 polar and acidic polar amino-acids with those containing 2 edges of the scale path, and 10 non-polar amino-acids with triangles containing 1 edge on the scale path is what comes first in mind.

The number of DNAs coding for a given amino-acid [I2] could be also seen as such a physical property. The model for dark nucleons leads to the vertebrate genetic code with correct numbers of DNAs coding for amino-acids. It is not however clear how to interpret DNA codons geometrically.

It however turns out that one can understand only the role of 60 codons in the icosahedral framework.

The treatment of the remaining 4 codons and of the well-known 21st and 22nd amino-acids requires the fusion of icosahedral code with tetrahedral code represented geometrically as fusion of icosahedron and tetrahedron along common face which has empty interior and is interpreted as "empty" amino-acid coded by stopping codons. In this manner one can satisfy the constraints on the Hamiltonian cycles, and construct explicitly the icosahedral Hamiltonian cycle as (4,8,8) cycle whose unique modification gives (4,11,7) icosa-tetrahedral cycle. Remarkably, two months after writing the first version of the article I learned that the data needed to calculate the Hamiltonian cycles can be found from web and that (4,8,8) cycle allows at least three realizations whereas the original candidate (3,10,7) allows no realization with symmetries but could do so with no symmetries.

2 Could Pythagoras have something to give for the modern musicology?

The ideas of Pythagorean school about music were strongly based on the number theory of that time. So called modern approaches tend to seem music scales as cultural phenomena. There are however many reasons to suspect that Pythagorean school might have been much nearer to truth.

2.1 Pythagoras and transition from rational numbers to algebraic numbers

Pythagoras was one the greatest ancient mathematicians. The prevailing belief at that was that the world can be described solely in terms rational numbers. During the times of Pythagoras the ancient mathematical consciousness had entered at the verge of a profound revolution: the time had become ripe for the discovery of algebraic numbers expanding rational numbers to an infinite series of algebraic extensions of rationals containing also rational multiples for finite number of algebraic numbers emerging as roots of polynomials with rational coefficients. Euclid introduces square root geometrically as length of the diagonal of square. In ancient India it was discover 800-500 BC, possibly much earlier. Unfortunately, the emergence of Christianity stopped the evolution of mathematics and new progress began at times of Newton when also reformation took place.

The well-known but story (good story but probably not true) tells that a pupil of Pythagoras demonstrated that the diagonal of unit square ($\sqrt{2}$) cannot be rational number and had to pay with his life for the discovery. Pythagoras himself encountered $\sqrt{2}$ through music theory. He asked what is the note exactly in the middle of the of the scale. Modern mathematician would answer half of octave corresponding to the frequency ratio $2^{1/2}$. Algebraic numbers did not however belong to the world of order of Pythagoras and he obtained to a non-satisfactory rational approximation of this number. This was very natural since only rational approximations of algebrais are possible in the experimental approach using only strings with rational number valued lengths. $\sqrt{2}$ represents the interval $C - F_{\#}$ known as tritone and this this interval was associated with devil and its use was denied also by church. Only after reformation $\sqrt{2}$ was accepted and this interval appears repeatedly in the compositions of Bach.

The amazing connections between evolution of mathematics and evolution of the religious beliefs inspires the question whether the evolution of consciousness could at basic level correspond to th evolution of the complexity of the number field behind the dynamics underlying consciousness. For instance, in TGD framework the vision about physics as generalized number theory allows one can to ask whether the mathematical evolution could have meant quite concretely the emergence of increasingly algebraic extensions of rationals for the coefficients of polynomials describing space-time surfaces serving as space-time correlates of consciousness.

2.2 Pythagoras and music

Pythagoras was both mathematician and experimentalist studying the world of musical experience experimentally. String instruments were his tool. The notion of frequency was not know at the time and length of vibrating part of string was the notion used. The experienced equivalence of notes differing by octave was known at that time and octave equivalence was understood as a fundamental symmetry of music manifesting itself as a scaling-by-2 symmetry for the length of a vibrating string.

Pythagoras developed 8 note scale CDEFGAHC (as a matter fact, 7 notes by octave equivalence) as we know as a combination of two scales EFGA and HCDE using octave equivalence and it was established as the official music scale. Pythagorean scale is expressed solely in terms of rational number valued ratios of the string length to that for the basic note of the scale (ratio of frequency to the fundamental).

Pythagorean scale (http://en.wikipedia.org/wiki/Music_theory, http://en.wikipedia.org/wiki/Music_and_mathematics) is expressed solely in terms of powers of the the ratio 3/2 for lengths of vibrating strings correspond to an interval known and complete fifth (C-G). The series of complete fifths (C-G-D-A...) known as progression by fifths gives very nearly 7 octaves but not quite: $(3/2)^{12} \simeq 128 + 1.75 = 2^7 + 1.745$. It would have been very natural to build 12-note scale as powers of rational (3/2) or by octave equivalence as powers of 3. The failure to close is very small but people with absolute ear experience the transponation of a melody to different key as dissonant since the frequency ratios do not remain quite same. At the time of Bach (Well tempered Klavier) the equal tempered scale obtained by diving the logarithmic scale to 12 equally long parts emerged and replacing powers of 3/2 with the 12 powers of algebraic number $2^{1/12}$ inside same octave even without octave equivalence emerged.

By octave equivalence Pythagorean scale means that all notes of the scale come in powers of 3 which strongly brings in mind 3-adicity. If one does not use octave equivalence when generalization of p-adicity to q-adicity with $q = 3/2$ is highly suggestive. q-adic numbers do not in general form number field, only an algebra.

Later more complex rational number based representations of scale using octave equivalence have been developed. The expression of the frequency ratios of the notes of the scale in terms of harmonic of fundamental modulo octave equivalence and involving only integers consisting of primes 2,3,5 is known as just intonation (http://en.wikipedia.org/wiki/Music_and_mathematics).

2.2.1 Music and Platonic solids

Pythagoras was also aware of a possible connection between music scales and Platonic solids. Pythagoras is claimed to have discovered tetrahedron, hexahedron (cube) and dodecahedron while octahedron and icosahedron would have been documented by greek mathematician Thaletus two hundred years later. The tetrachord and was assigned with tetrahedron and one and imagined that Pythagorean scale could have been assigned with pair of tetrahedra somehow - cube or octahedron which comes in mind. Note that this would require that basic note and its octave should be regarded as different notes.

These attempts inspire the question whether the mapping music scales to the vertices of Platonic solids could provide insights about music experience. One can also ask whether there might be a mapping of music understood as melodies and chords in some scale to the geometries defined by Platonic solids.

1. Since 12-note scale is used in practically all classical western music and even in atonal music based on 12-note scale, the natural question is whether 12-note scale could be mapped to a connected, closed, non-self-intersecting path on icosahedron going through all 12 vertices and consisting of edges only. Closedness would mean that base note and its octave are identified by octave equivalence.

2. This mathematical problem is well-known and curves of this kind are known as Hamilton cycles and can be defined for any combinatorial structure defined by vertices and faces. Hamilton proved that Hamiltonian cycles (possibly identifiable as 20-note scale) at dodecahedron is unique module rotations and reflection leaving dodecahedron invariant. Also in the case of tetrahedron and cube the Hamiltonian cycle is unique.

3. For octahedron and icosahedron this is not the case [A3] and there are both cycles containing only faces with at least 1 edge of the path and also cycles containing no faces containing no edges of the path. Numerical experimentation is rather straightforward manner to determine Hamiltonian cycles and $H = 2^{10} = 1024$ cycles can be found. The number of topologically non-equivalent cycles (not transformable to each other by the isometries of icosahedron) is factor of this number. The group of orientation preserving isometries of icosahedron is the alternating group A_5 of 60 even permutations of five letters. The full group of isometries is $G = A_5 \times Z_2$ containing $N = 120$ elements.

4. Some subgroup of G leaves given path invariant and its order must be factor M of N so that topological equivalence class of cycles contains $R = N/M$ elements. The number of topologically non-equivalent cycles in given class with $H(top)$ elements is $N_{tot} = H(top)/R$ so that Rmust be a factor of $H(top)$.

Before continuing it is good so summarize the geometry of icosahedron shortly. There are 20 faces which are triangles, 12 vertices, and 30 edges. From each vertex 5 edges. Therefore the construction of Hamiltonian cycles means that at each vertex on path one must select between four options edges since one cannot return back. This gives $4^{12} = 2^{24} \sim 1.6 \times 10^7$ alternatives to be considered. Therefore the numerical search should be relatively easy. Keeping account of the points already traversed and not allowing self intersections, the actual number of choices is reduced. The construction requires labeling of the vertices of the icosahedron by integers 1,...,12 in some manner and defining 12×12 matrix $A(i,j)$ whose element equals to 1 if vertices are neighbours and 0 if not. Only the edges for with $A(i,j) = 1$ holds true are allowed on the path. A concrete representation of icosahedron as a collection of triangles in plane with suitable identifications of certain edges is needed. This helps also to visualize the classification of triangles to three types discussed below. This can be found in the Wikipedia article.

2.2.2 Numbers of different triangles as characterizers of harmony

A possible interpretation for topologically non-equivalent paths is as different notions of harmony.

1. Proceeding in Pythagorean spirit, the neighboring points would naturally correspond to progression by fifths - that is scalings by powers of 3/2 or in equal temperated scale by powers of $2^{7/12}$. This would mean that two subsequent vertices would correspond to quint.

2. The twenty triangles of the icosahedron would naturally correspond to 3-chords. Triangles can contain either 0, 1, or 1 edges of the 12-edge scale path. The triangle containing 3 edges is not possible since it would reside on a self-intersecting path. A triangle containing one edge of path the chord would contain quint which suggest a chord containing basic note, quint and minor or major third. The triangle containing two edges would contain subsequent quints - CDG is one possible example by octave equivalence. If the triangle contains no edges of the path one can say that the chord contains no quints.

   The numbers of triangles classified according to the number of path edges contained by them serves as the first classification criterion for a given harmony characterized by the Hamiltonian cycle (note that one cannot exclude the possibly of non-closed paths since Pythagorean construction of the scale by quints does not yield quite precisely octave as outcome) (see Fig. 2.2.2).

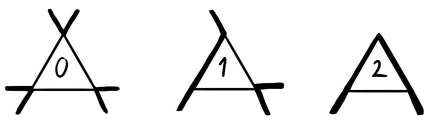

Figure 1: There 3 different types of triangles characterized by the number of edges contained by them. This predicts chords with 0, 1 or 2 quints.

Consider now the situation in more detail.

1. The topologically equivalent cycles must have same numbers of faces containing 0, 1, or 2 edges of the Hamiltonian path since isometries do not change these numbers. Let us denotes these numbers by n_0, n_1 and n_2. The total number of faces is 20 so that one has

$$n_0 + n_1 + n_2 = 20 \ .$$

Furthermore, each of the 12 edges on the path is contained by two faces so that by summing over the numbers of edges associated with the faces one obtains twice the number of edges:

$$0 \times n_0 + 1 \times n_1 + 2 \times n_2 = 2 \times 12 = 24 \ .$$

From these constraints one can solve n_0 and n_1 as function of n_2:

$$\begin{array}{ll} n_0 = n_2 - 4 \ , & n_2 \geq 4 \ , \\ n_1 = 24 - 2n_2 \ , & n_2 \leq 12 \ . \end{array}$$

If these integers characterize the topological equivalence completely and if the allowed combinations are realized, one would have 12-4=8 topologically nonequivalent paths. The actual number is $N_{tot} = 2^k$, $k \geq 7$, so that the integers cannot characterize the topology of the path completely.

2. The number of Hamiltonian cycles on icosahedron is known to be 2560 [A1]. Numerical calculations [A2] (`http://mathoverflow.net/questions/37788/why-are-there-1024-hamiltonian-cycles-on-an-icos` shows that the number of Hamiltonian cyclesw with one edge fixed is $2^{10} = 1024$. Here one regards cycles with different internal orientation as different. This would mean that the sum over the numbers $N(n_2)$ if cycles associated with differ values of n_2 satisfies

$$\sum_{n_2=4}^{12} \sum_i N(n_2, i)) = 2^{10} \ .$$

$N(n_2, i)$ is the number of paths of given topology with fixed n_2. The numbers $N(n_2, i)$ are integers which are factors of $N = 120$ of the order of the isometry group of the icosahedron. The average of $N(n_2, i)$ is $2^7 = 128$.

2.2.3 Additional topological invariants characterizing the notion of harmony

The interpretation of amino-acids in terms of 20 triangles of icosahedron interpreted as allowed chords for a given notion of harmony leads to a unique identification of thee integers n_i as $(n_0, n_1, n_2) = (3, 10, 7)$. The attempt to interpret this "biological harmony" leads to the identification of additional topological invariants characterizing the notion of harmony. It will be assumed that edges correspond to quints. If they would correspond to half-step the chords would contains 0, 1, or 2 subsequent half-intervals which does not conform with the usual views about harmony. In Pythagorean scale quint corresponds to 3/2 and in equal tempered scale quint corresponds to the algebraic number number $2^{7/12}$.

Above the attention was paid to the properties of the triangles in relation to the Hamiltonian cycle. One can consider also the properties of the edges of the cycle in relation to the two neighboring triangles containing it (see Fig. 2.2.3). Restrict first the attention to the biological harmony characterized by $(n_0, n_1, n_2) = (3, 10, 7)$.

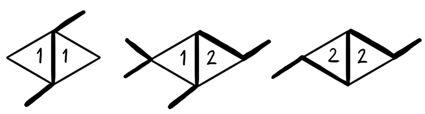

Figure 2: The edge of the cycle belongs to 2 triangles, which as chords can correspond to 1 resp.2 , 1 resp. 1 and 2 resp. 2 quints.

1. Everyone of the 12 quints $C - G$, $C_\# - G_\#$,... would be contained to neighboring triangles tht is 3-chords containing at least one quint. Denote by p_{12}, p_{11} resp. p_{22} denote the number of edges

shared by 1-quint triangle and 2-quint triangle, by 2 1-quint triangles, resp. 2 2-quint triangles. Besides $p_{ij} \geq 0$ one has

$$\sum p_{ij} = 12 \ .$$

since the cycle contains 12 edges. There are $p_{12}+2p_{11} = n_1$ 1-quint triangles and $(p_{12}+2p_{22})/2 = n_2$ 2-quint triangles (note double counting responsible for division by two). Altogether this gives

$$\begin{array}{l} p_{22} = 12 - p_{11} - p_{22} \ , \\ p_{22} = p_{11} + n_2 - \frac{n_1}{2} \ , \\ p_{22} = n_2 - \frac{p_{12}}{2} \ . \end{array}$$

2. These three Diophantine equations are for integers and would allow for real numbers only single solution and for integers it in the generic case there are no solutions at all. Situation changes if the equations are not independent which can happen if the integers n_i satisfy additional conditions. By subtracting first and second and second and third equation from each other one obtains the consistency condition

$$n_1 = 24 - 2n_2 \ .$$

This condition is however second of the conditions derived earlier so that only two equations, say the first two ones, are independent.

$$\begin{array}{l} p_{22} = p_{11} + n_2 - \frac{n_1}{2} \ , \\ p_{22} = n_2 - \frac{p_{12}}{2} \ . \end{array}$$

giving

$$\begin{array}{l} p_{11} = (n_1 - p_{12})/2 \ , \\ p_{22} = p_{11} + n_2 - \frac{n_1}{2} = n_2 - \frac{p_{12}}{2} \ . \end{array} \quad .$$

One must have $0 \leq p_{ij} \leq 12$ and $p_{12} \leq n_1$ from $p_{11} = (n_1 - p_{12})/2$. Here one has $p_{12} \in \{0, 2, ...Min\{12, 2n_2, n_1\}$ so that $Min\{7, n_2 + 1, [n_1/2] + 1\}$ solutions are possible. The condition that the cycle has no self-intersections can forbid some of the solutions.

3. The first guess for the "biological harmony" possibly associated with amino-acids would be $(n_0, n_1, n_2) = (3, 10, 7)$: this if one neglects the presence of 21st and 22th amino-acid also appearing in proteins. It turns out that a more feasible solution fuses tetrahedral code and icosahedral codes with $(n_0, n_1, n_2) = (4, 8, 8)$ giving $(n_0, n_1, n_2) = (4, 11, 7)$ for icosatetrahedral code.

   For instance, $(n_0, n_1, n_2) = (3, 10, 7)$ would give $p_{12} \in \{0, 2, 4, 6, 8, 10\}$, $p_{11} \in \{5, 4, 3, 2, 1, 0\}$, $p_{22} \in \{7, 6, 5, 4, 3, 2\}$ so that one has 6 alternative solutions to these conditions labelled by p_{12}. The number of neighboring triangles containing single quint is even number in the range $[0, 10]$: this brings in mind the possibility that the neighboring single quint triangles correspond to major-minor pairs. Clearly, the integer p_{12} is second topological invariant characterizing harmony.

2.2.4 Distribution of different types of edges

Also the distribution of the 12 edges to these 3-types is an invariant characterizing the shape of the curve and thus harmony as isometric invariant (see Fig. 2.2.4).

1. p_{12} 1-1 edges can be chosen in

$$N(1-1, p_{12}) = \binom{12}{p_{12}}$$

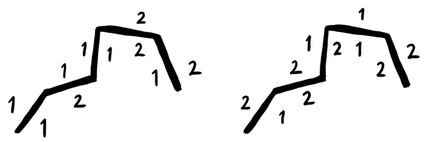

Figure 3: Also the distributions for three types of edges matter.

manners and 1-2 edges in

$$N(1-2, p_{12}) = \left(\begin{array}{c} 12 - p_{12} \\ p_{12} \end{array} \right)$$

manners. The remaining 2-2 edges can be chosen only in one manner. This gives altogether

$$N(p_{12}) = N(1-1, p_{12}) \times N(1-2, p_{12})$$

manners for given value of p_{12}.

To summarize, one obtains large number of notions of harmony are possible although one cannot expect that the absence of self-intersections does not allow all topologies for the cycle.

2.3 Would you come to icosadisco with me?

This map would allow one-to-one map of the notes of any music piece using icosahedral geometry. If octave equivalence is assumed, a given note would be mapped to a fixed vertex of icosahedron at which lamp is turned on and also to the wavelength of the light in question since visible light spans an octave. Chords would correspond to the turning on of lights for a group of icosahedral points. Icosahedrons with size scaled up by two could correspond to octave hierarchy: for practical purposes logarithmic scale implying that icosahedrons have same distance would be natural as in the case of music experience since piano spans 7 octaves and human ear can hear 10 octaves. Church would nowadays allow icosadiscos to use also half octaves to amplify further the audiovisual inferno effect so characteristic for discos. One could also try to realize special effects like glissandos, vibratos and tremolos.

3 Connection between music molecular biology?

Music affects directly emotions, and consciousness is one aspect of being living. This raises the question whether the Platonic geometries might have something to do with basic building bricks of life and with genetic code.

3.1 Could amino-acids correspond to 3-chords of icosahedral harmony?

The number of amino-acids is 20 and same as the number of triangular faces of icosahedron and the vertices of dodecahedron. I have considered the possibility that the faces of icosahedron could correspond to amino-acids [K1]. Combined with the idea about connection between music scale and icosahedron this inspires the following consideration.

1. For a proper choice of the mapping of the 12-note scale to the surface of icosahedron the 20 triangles could correspond to 20 amino-acids analogous to 3-chords and that the 3 types of 3-chords could correspond to 3 different classes of amino-acids. One can of course consider also the mapping of amino-acids to a unique sequence of 20 vertices of dodecahedron representing 20-note scale or 20-chord scale and replacement of the 3-chords defining the harmony with 12 5-chords.

2. Amino-acids are characterized by the non-constant side chain and these can be classified to three categories: basic polar, non-polar, and polar (http://en.wikipedia.org/wiki/Amino-acids). The numbers of amino-acids in these classes are $a_0 = 3$, $a_1 = 10$, $a_2 = 7$. Could these classes correspond to the numbers n_i characterizing partially some topological equivalence classes of Hamiltonian paths in icosahedron? There is indeed a candidate: $a_0 = n_0 = 3$, $a_1 = n_1 = 10$, $a_2 = n_2 = 7$ satisfies the conditions discussed above. 3 basic polar amino-acids would correspond to the triangles with no edges on the Hamiltonian cycle, 10 non-polar amino-acids to triangles containing one edge, and 7 acidic polar and polar amino-acids to those containing two edges. One can criticize the combination of polar and acidic polar amino-acids in the same class. One can also classify amino-acids to positively charged (3), negatively charged (2) and neutral (15) ones. In this case the condition is however not satisfied. Thus the proposal survives the first test - assuming of a course that these Hamiltonian cycles exist! This has not been proven and would require numerical calculations.

3. As found Hamiltonian paths have also other topological characteristics and they could correspond to physical characteristics and it would be interesting to see what theyh are. To proceed further one should find the total number of the Hamiltonian paths with $n_2 = 7$ and identify the isometries of different topological equivalence class having $n_2 = 7$.

Amino-acid sequences would correspond to sequences of 3-chords. The translation of mRNA of gene to amino-acid sequence would be analogous to the playing of a record. The ribosome complex would be the record player, the amino-acid sequence would be the music, and mRNA would be the record. Hence genes would define a collection of records characterizing the organism.

3.2 Can one understand genetic code?

What remains open is the interpretation of genetic code [I2]. DNA triplets would correspond naturally to triangles but why their number is 64 instead of 20. They would be obviously the analogs of written notes: why several notes would correspond to the same chord?

1. Could different DNA triplets coding for the same amino-acid correspond to various octaves of the chord? The most natural expectation would be that the number of octaves so that one would have 3 DNAs would code single amino-acid and stopping codon would correspond to 4 DNAs. It is difficult to understand why some 3-chords could correspond to 6 octaves and one of them only one.

2. Could the degeneracy correspond to the ordering of the notes of the 3-chord? For the 3-chords there are 6 general orderings and 3 cyclic orderings modulo octave equivalence and characterizing by the choice of the lowest note. The simplest assumption would be that the allowed orderings - degeneracies - are characterized by a subgroup of the cyclic group S_3 yielding the allowed permutations of the notes of the chord. The subgroup orders for S_3 are 1,2,3, and 6. The allowed degeneracies are 6,4,3,2, and 1 so that this identification fails for $D = 4$.

3. Could the different correspondences between DNA codons and amino-acids correspond to the different topological equivalence classes of $n_2 = 7$ Hamiltonian cycles. This does not seem to be the case. The number of different DNA-amino-acid correspondences obtained by choosing one representative from the set of DNAs coding for a given amino-acid (and not stopping sign) is the product of the numbers $D(a_i)$ coding amino-acid a_i. From the table below this number is given by $6^3 \times 4^5 \times 3^1 \times 2^9 \times 1^2 = 3^4 \times 2^{21}$ and clearly much larger than $N = 2^{10}$.

4. Could the different codons coding for codon code for some additional information so that amino-acids would in some aspect differ from each other although they are chemically identical? Here the magnetic body of amino-acid is a natural candidate. This would suggest that the folding pattern of the protein depends on what DNA sequence codes it. This information might be analogous to the information contained by notes besides the frequencies. Durations of notes corresponds is the most important information of this kind: the only candidate for this kind of information is the value of $h_{eff} = n \times h$ associated with the amino-acid magnetic body determining its size scale. Magnetic fields strength could be also code by DNA codon besides amino-acid.

d	6	4	3	2	1
N	3	5	2	9	2

Table 1. The number of amino acids N associated with a given degeneracy d telling the number of DNA triplets mapped to the amino acid in the genetic code. The degeneracies are always smaller than 7 as predicted by the proposed explanation of the Genetic Code.

Second question concerns genetic code itself. Could the DNA degeneracies $D(a_i)$ (number of DNAs coding for amino-acid a_i) be understood group theoretically in terms of icosahedral geometry? The triangles of the icosahedron are mapped the triangles under the isometries.

1. One can start by looking the Table 1 for the genetic code telling the number $N(d)$ of amino-acids coded by d DNA codons. One finds that one can divide DNAs to three groups containing $n = 20$, $n = 20$, resp. $n = 21$ codons.

   (a) There are 3 amino-acids codes by 6 codons and 2 amino-acids coded by 1 DNA: $3\times 6+2\times 1 = 20$ codons altogether.
   Note: One could also consider 1 amino-acid coded by 2 codons instead of 2 coded by 1 codon $3 \times 6 + 1 \times 2 = 20$.

   (b) There are 5 amino-acids coded by 4 codons making $5 \times 4 = 20$ codons altogether.

   (c) There are 9 amino-acids coded by 2 codons and 1 by 3 codons making $9 \times 2 + 1 \times 3 = 21$ codons.
   Note: One could also consider the decomposition $8 \times 2 + 2 \times 1 + 1 \times 3 = 21$ codons implied if 1 amino-acid is coded by 2 codons in the first group.

This makes 61 codons. There are however 64 codons and 3 codons code for stopping of the translation counted as "empty" amino-acid in the table.

1. This would suggest the division to 60 + 4 codons. The identification of additional 4 codons and corresponding amino-acids is not so straightforward as one might first think. 3 of the 4 additional codons could code for "empty" amino-acid (Ile) and 1 of them to Ile (empty amino-acid).

2. What suggests itself strongly is a decomposition of codons in 3 different manners. 3 groups of 6 codons plus 2 groups of 1 codon (1 group of 2 codons), 5 groups of 4 codons, and 10 groups of 2 codons (9 groups of 2 codons plus plus 2 groups of 1 codon).

This kind of decompositions are induced by the action on the triangles of icosahedron by thee subgroups of the isometry group $A_5 \times Z_2$ of the icosahedron having $120 = 2 \times 2 \times 2 \times 2 \times 3 \times 5$ elements and subgroups for which number of elements can be any divisor of the order. The orbit associated with a subgroup with n elements has at most n triangles at its orbit. This allows immediately to deduce the values of n possibly explaining the genetic code in the proposed manner.

1. The 3 amino-acids coded by 6 codons must correspond to $n = 6$. This subgroup must have also two 1-element orbits (1 2-element orbit): in other words, 2 triangles must be its fixed points (form its orbit).

   (a) The non-abelian group S_3 permuting the vertices of is the first candidate for the subgroup in question. The triangles at the opposite sides of the icosahedron remain invariant under these permutations. S_3 however has two orbit consisting of 3 triangles which are "wall neighbours" of the triangles which remains fixed.

   (b) Second candidate is the abelian group $\tilde{Z}_2 \times Z_3$. Here Z_3 permutes the vertices of triangle and $\tilde{Z}_2$ is generated by a reflection of the triangle to opposite side of icosahedron followed by a rotation by π. This group has 3 orbits consisting of 6 triangles and 1 orbit consisting of 2 triangles (the triangles at opposite side of icosahedron). This group seems to be the only working candidate for the subgroup in question.

2. The 5 amino-acids coded by 4 codons must correspond to $n = 4$ and therefore to $\tilde{Z}_2 \times Z_2$. This is indeed subgroup of icosahedral group which permutes triangles at the vertices of inscribed tetrahedron. Now all orbits contain 4 triangles and one must have 5 orbits, which are obtained by acting on the 5 triangles emanating from a given vertex. Note that also Z_5 is subgroup of icosahedral group: this would give a variant of code with 4 amino-acids coded by 5 codons if it were possible to satisfy additional consistency conditions.

3. Consider next the group consisting of 9 amino-acids coded by 2 codons and Ile ("empty" amino-acid) coded by 3 codons. Since only the $\tilde{Z}_2 \times Z_3$ option works, this leaves 9 amino-acids coded by 2 codons and 2 amino-acids coded by 1 codon. The subgroup must correspond to $n = 2$ and thus Z_2 acting on fixed triangle and leaving it and its $\tilde{Z}_2$ image invariant. One has 9 2-triangle orbits and two single triangle orbits corresponding to the triangles at the opposite sides of the icosahedron. The 9 amino-acids coded by 2 codons are all real or 8 of them are real and 1 corresponds to "empty amino-acid" coded by two codons.

3-element orbits are lacking and this forces to consider a fusion of of icosahedral code with tetrahedral code having common "empty-acid" - common triangle of icosahedron and tetrahedron) coded by 2 icosahedral codons and 1 tetrahedral codon. Ile would be coded by 3 codons assignable to the orbit of Z_3 subgroup of tetrahedral symmetry group S_3 and would be associated with the tetrahedron. This would predict 2 additional amino-acids which could be understood by taking into account 21st and 22nd amino-acid (Sec and Pyl [I1]).

The Hamiltonian cycle is not explicitly involved with the proposed argument. Some property of the cycle respected by the allowed isometries might bring in this dependence. In Pythagorean spirit one might ask whether the allowed isometries could leave the Hamiltonian cycle invariant but move the vertices along it and induce a mapping of faces to each other.

The amino-acid triangle at given orbit cannot be chosen freely. The choices of amino-acid triangles associated with the three groups of 20 DNAs must be different and this gives geometric conditions for the choices of the three subgroups and one can hope that the assignment of amino-acid toa given triangle is fixed about from rotational symmetries.

3.2.1 Does the understanding of stopping codons and 21st and 22nd amino-acids require fusion of tetrahedral and icosahedral codes?

Several questions remain. Could one also understand the additional 4 DNA codons? Could one understand also how one of them codes amino-acid (Ile) instead of stopping codon? Can one related additional codons to music?

3.2.2 Attachment of tetrahedron to icosahedron as extension of isosahedral code

The attachment of tetrahedron to icosahedron (see Fig. 3.2.2) allows to understanding both stopping codons and "empty" amino-acid as well as the 21th and 22nd amino-acids geometrically.

1. Something is clearly added to the geometric structure, when 4 additional DNA codons are brought in. They could represent orbits of faces of Platonic solid with 4 faces. The four faces are most naturally triangles and actually must be so since tetrahedron is the only Platonic solid having 4 faces and its faces are indeed triangles. Tetrahedron has symmetry group S_3 containing Z_3 and Z_2as subgroups. Z_3 leaves one of the tetrahedral triangles invariant so that one has two orbits consisting of 1 and 3 triangles respectively.

2. One amino-acid is coded by 3 rather than only 2 codons. One can indeed understand this symmetry breaking geometrically. If the tetrahedron is attached on icosahedron along one of its triangular faces and this icosahedral face corresponds to amino-acid (Ile) belonging to a 2-triangle orbit under icosahedral Z_2 one obtains 3 codons coding for Ile! I have actually proposed earlier a model involving attachment of tetrahedrons to icosahedron as a model for genetic code.

3. Tetrahedron should bring in three additional amino-acids. "Empty" amino-acid could correspond to either one of them or to the common base triangle which is indeed geometrically in unique

position. One could even demand that this triangle is "empty" so that tetraicosahedron would be non-singular continuous manifold. This would mean exchange of the roles of ile and empty amino-acid both being coded by three codons. The 3-triangle orbit outside the icosahedron would correspond to Ile and base triangle to empty amino-acid or vice versa. Base triangle would be coded by 1 tetrahedral codon plus 2 icosahedral codons.

One of the outsider triangles corresponds to "empty" amino-acid or ile but two other triangles to two new exotic amino-acids. In some species there indeed are 21st and 22nd amino-acids (selenocysteine and pyrrolysine, `http://en.wikipedia.org/wiki/Amino_acid`) with sulphur replaced with selene. This modification does not change the polarity properties of cys and lys: cys is non-polar and lys basic polar implying $(n_0, n_1, n_2) \rightarrow (4, 11, 7)$.

4. The naive assumption $(n_0, n_1, n_2) = (3, 10, 7)$ before the modification need not be correct and the numerous futile attempts to construct this cycle and the argument below for construction of icosahedral Hamilton cycle suggests that $(n_0, n_1, n_2) = (4, 8, 8)$ is more feasible option.

5. The two outsider tetrahedral triangles could correspond to the orbits of Z_2 subgroup of S_3 acting as reflection with respect to media of the bottom triangle. Outside faces form orbits consisting of 1 triangle and 2-triangles. These orbits could correspond to 21st and 22nd amino-acids coded by 1 and 2 exotic variants of stopping codons respectively.

   The 2 exotic amino-acids are however coded by codons which are usually interpreted as stopping codons. Something must however distinguish between standard and exotic codings. Is it "context" giving different meaning for codons and perhaps characterized by different magnetic bodies of codons?

`/Users/mattipitkanen/Desktop/tgd/figures/tetraicosahedron.jpg`

Figure 4: tetraicosahedron is obtained by attaching tetrahedron along one of its faces to icosahedron. The resulting structure is topological manifold if the common face is replaced with empty set and it is natural to identify it as "empty" amino-acid.

The base triangle is counted neither as amino-acid nor as a part of the "tetraicosahedron". The interpretation is natural since the resulting topological structure is non-singular and defines a continuous albeit not differentiable manifold topology (sphere).

3.2.3 How the icosahedral Hamiltonian cycle is modified?

The properties of exotic amino-acids give constraints on how the modification of the Hamiltonian cycle should be carried out. The naive expectation that the outer triangles of added tetrahedron correspond to "empty" amino-acid and 2 exotic amino-acids is not correct. A more appropriate interpretation is as a fusion of icosahedral and tetrahedral codes having common "empty amino-acid" coded 2 icosahedral and 1 tetrahedral 1 stopping codons respectively and obtained by gluing these Platonic solids together along the triangle representing the "empty" amino-acid. That the common triangle corresponds to "empty" amino-acid means geometrically that its interior is not included so that the resulting structure is continuous manifold having topology of sphere.

Consider now the detailed construction.

1. One should be able to modify the icosahedral Hamiltonian cycle so that the numbers (n_0, n_1, n_2) charactering icosahedral cycle change so that they conform with the properties of the two exotic amino-acids. Selenocystein (Sec) is nonpolar like cys and pyrrolysine (Pyl) basic polar like Lys so that $(4, 11, 7)$ seems to be the correct characterization for the extended system. One must have $(n_0, n_1, n_2) \to (4, 11, 7)$.

2. One must visit the additional vertex, which means the replacement of one edge from the base triangle with wedge visiting the additional vertex. There are several cases to be considered depending on whether the base triangle is 1-quint triangle or 2-quint triangle, and what is the type of the edge replaced with wedge. One can even consider the possibility that the modified cycle does not remain closed.

   If the icosahedral cycle has $(n_0, n_1, n_2) = (3, 10, 7)$, the value of n_2 is not changed in the construction. For a closed cycle edge is replaced with wedge and the only manner to preserve the value of n_2 is that the process producing 1 tetrahedral 2-quint triangle transforms 1 icosahedral 2-quint triangle identified as base triangle to 1-quint triangle. If the replaced edge of base triangle is of type 2-1, one has $n_1 \to n_1 + 1$ since one icosahedral 1-quint triangle disappears and 2 tetrahedral ones appear. Icosahedral n_0 increases by 1 units. Hence the condition $(3, 10, 7) \to (4, 11, 7)$ would be met. It however seems that $(4, 8, 8)$ is more promising starting cycle as the argument below shows (see Fig. 3.2.3).

3. The number options is at most the number n_2 of 2-quint triangles serving as candidates for "empty" amino-acid. An additional condition comes from the requirement that replaced edge is of type 2-1.

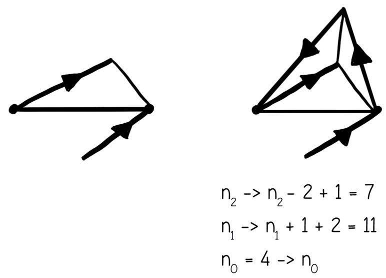

Figure 5: The modification of $(4, 4, 8)$ icosahedral Hamiltonian cycle consistent with the constraints that icosatetrahedral cycle corresponds to $(4, 11, 7)$ consistent the classification of amino-acids in three classes.

3.2.4 Direct construction of Hamiltonian cycle corresponding to bioharmony

Consider bio-harmony as an example about Hamiltonian cycle taking seriously the extension of the genetic code. I have made very many unsuccessful triangles starting from the assumption that icosahedral cycle satisfies $(n_0, n_1, n_2) = (3, 10, 7)$, and the following proposal starts from different icosahedral cycle. The following is just a trial, which should be checked by a direct calculation.

1. The most obvious guess for the cycle to be modified to cycle at tetraicosahedron having $(n_0, n_1, n_2) = (4, 11, 7)$ (the triangle corresponding to "empty" amino-acid is not counted) is $(n_1, n_2, n_3) = (3, 10, 7)$. I have not found cycle with these characteristics.

2. It seems however possible to find cycle with $(n_1, n_2, n_3) = (4, 8, 8)$. From this can obtain the desired kind of extended cycle if the "empty" triangle is 2-quint triangle and the edge replaced with the wedge is of type 2-2. The replacement of icosahedral edge eliminates two icosahedral 2-quint triangles and generates 1 tetrahedral 2-quint triangle giving $n_2 \to n_2 - 2 + 1 = n_2 - 1 = 7$. The disappearance of the icosahedral edge generates two icosahedral 1-quint triangles of which second one corresponds to empty amino-acid and is not counted and 2 tetrahedral 1-quint triangles giving $n_1 \to n_1 + 3 = 11$.

Figure 3.2.4 below represents the construction of cycle $(4, 8, 8,)$. The icosahedron is constructed from regions $P(I)$ glued to the triangle t along one edge each. The arrows indicate that the one pair of edges of type 1 and 2, 1 and 3 and 3 and 2 are identified. Also the long edges I of T are identified with pairs of subsequent edges of $P(I)$ as the arrows indicate.

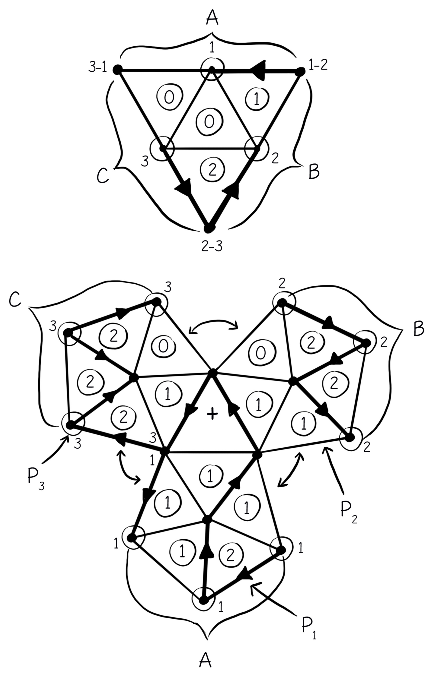

Figure 6: A proposal for a Hamilton cycle realizing bio-harmony $(n_1, n_2, n_3) = (4, 8, 8)$ allowing extension to cycle $(3, 11, 7)$ on tetraicosahedron. Circled "0", "1" and "2" indicates whether a given small triangle is 0-, 1-, or 2-quint triangle. It is relatively easy to verify that the condition $(n_1, n_2, n_3) = (4, 8, 8)$ for bio-harmony is satisfied.

3.2.5 Stopping codons and music

What could be the interpretation of the attached tetrahedron in terms of music harmony?

The attachment of tetrahedron means addition of an additional note to the 12-note scale. The scale constructed in Pythagorean spirit identifying quint as scaling by 3/2 contains the 12th note as scaling by $(3/2)^{12}$ of the basic frequency modulo octave equivalence. This is slightly more than scaling by 2^7 so that exact octave is not obtained. The attempt to solve this problem has lead to scales in which one allows a pair of notes with a very small interval between them - say $G_\#$ and A_b being regarded as different notes.

This suggests that the outsider vertex of the attached tetrahedron corresponds to a note very near to some note of the 12-note scale. Which note is in question depends on which of the 10 1-quint triangles is chosen as the base triangle. This is expected to imply additional refinements to the notion of bio-harmony. 2 or three additional 3-chords emerge depending on whether empty amino-acid is interpreted as a real chord.

This could also allow to understand why it is natural to have non-closed Hamiltonian path on tetra-icosahedron. Octave equivalence requiring closedness indeed fails in Pythagorean scale.

3.2.6 Geometric description of DNA-amino-acid correspondence

The mathematical structure which suggests itself is already familiar from some earlier attempts to understand genetic code [K2]. For icosahedral part of code one would have a discrete bundle structure with 20 amino-acids defining the base space and codons coding the amino-acid forming the fiber. The number of points in the fiber above based point depends on base point and is the number of codons coding the corresponding amino-acid. A discrete variant of singular fiber bundle structure would be in question.

Forgetting for a moment the 4 troublesome codons, the bundle would be the union of the orbits associated with groups S_3, Z_4 and Z_2 of icosahedral group, and the base would consist of 20 amino-acids, one for each orbit. The point of orbit must be selected so that the selections for orbits of two different groups are different.

The addition of the additional codons, "empty" amino-acid and two exotic amino-acids would mean gluing of tetrahedron along one of its faces to icosahedron. This would induce extension of the singular bundle like structure. To each of the new faces one would attach the orbit of triangles representing the codons coding for the corresponding amino-acid.

To sum up, in its strongest form the model makes several purely mathematical predictions, which could easily kill it.

1. The identification of the 3-chords assignable to the triangles of the icosahedron.
2. The existence of $n_2 = 7$ Hamiltonian cycle requiring however the lumping of acidic polar and polar amino-acids in the same class.
3. The possibility to select one representative of amino-acid from each group orbit (S_3, Z_4, Z_2) such that all amino-acids are different is non-trivial and one should prove that this is possible. One must decompose the set of amino-acids to orbits in 3 different manners (S_3, Z_4, Z_2) into orbits and select for each decomposition 1 representative from each orbit such that the selection contains all 20 triangles.

3.3 How could one construct the Hamiltonian cycles on icosahedron with a minimal computational work?

Although the construction of Hamiltonian cycles is known to be an NP hard problem for a general graph, one can hope that in case of Platonic solids having high symmetries, a direct construction instead of straightforward numerical search might work. The following is a proposal for how one might proceed. It relies on paper model for icosahedron.

1. The basic observation about one can get convinced by using paper model is following. One can decompose the surface of icosahedron to three regions $P(I)$, $I = 1, 2, 3$, with pentagonal boundary and containing 5 triangles emanating from center vertex plus one big triangle T containing 4 pentagonal triangles and one lonely small triangle t opposite to it. These 5 regions span the surface of icosahedron. There is clearly a symmetry breaking and there is great temptation to assume that t corresponds to the triangle along which the tetrahedron is glued to the icosahedron in the model of genetic code realizing the modification of icosahedral cycle for which the first guess is $(3, 7, 10)$ but for which also $(4, 8, 8)$ can be considered.
2. The Hamiltonian cycle must visit at the centers of each $P(I)$: one enters pentagonal region $P(I)$, $I = 1, 2, 3$ along one of the five interior edges beginning at pentagonal vertex $a_{I,i}$, $i = 1, .., 5$ and leaves it along second edge ending at vertex $b_{I,j}$, $j \neq 5$. One can call these edges interior edges.

The edges at boundaries of $P(I)$ can be called boundary edges. Interior edge can correspond to $|i-j| = 0, 1$ or $i-j > 1$. For $|i-j| = 1$ the interior edge gives rise to 2-quint triangle. For $i-j = 0$ there is no boundary edge after $b_(I, j)$.

3. Pentagonal boundary edges come in three types. 2 of them are shared with T, 1 with t opposite to it, and 2 with another pentagonal region $P(I)$. One can label $P(i)$ in such a manner that the $P(I)$ shares two boundary edges with $P(I+1)$.

   The boundary edges of small and big triangle are boundary edges of the 3 pentagonal regions so that they are not counted separately.

4. One can assume that the cycles begins from a vertex of T. Since the cycle is closed it returns back to this vertex. The last edge is either at the boundary of T or goes through one or two edges of the small interior triangle of T so that this triangle is either 0-,1- or 2-quint triangle.

   t can be 0-, 1-, or 2-quint triangle.

5. The total number of the interior edges inside the 3 pentagonal regions is $3 \times 2 = 6$ so that 6 remaining edges must be boundary edges associated with $P(I)$ and interior edges of T: otherwise one would visit some pentagonal center twice and self-intersection would occur. The boundary edges associated with t and T are boundary edges of $P(I)$, $I = 1, 2, 3$

6. At the vertex $b_(I, j)$ of pentagonal region one must turn right or left and move along the boundary edge. One can move at most $n_I = 4-j$ boundary edges along the pentagonal boundary in clockwise direction and $n_I = j-2$ edges in counterclockwise direction (clockwise is the direction in which the index labelling 5 vertices grows). The maximum number of boundary edges is 3 and obtained for $j - i \pm 1$.

7. The condition $\sum n_I + n(T) = 6$, where $n(T) = 1, 2$ is the number of interior edges of T, holds true so that one has $\sum n(I) \equiv n_{tot} \in \{4, 5\}$. The numbers and types (shared with pentagon, T, or t) of the boundary edges of $P(I)$, the differences $\Delta(I) = j_I - i_I$, the number of edges in t and the number of interior edges of T characterize the Hamiltonian cycle besides the condition that it is closed. The closedness condition seems possible to satisfy. One must enter big triangle through one of the vertices of T and this vertex is uniqely determined once the third pentagon is fixed. One can therefore hope that the construction gives directly all the Hamiltonian cycles with relatively small amount of failed attempts, certainly dramatically smaller than $n = 2^{24} \sim 10^7$ of blind and mostly un-succesful trials.

8. Each $P(I)$ containing boundary edges gives rise to least 2 2-quint triangles associated with $b_I(I)$ and a_{I+1}.

   If all 3 $P(I)$ have $|i-j| > 1$, one has $n_2 = 3 \times 2 = 6$. The contribution of regions $P(I)$ is larger if some pentagon interiors have $|\Delta(I)| = |j(I) - i(I)| = 1$. $|j(I) - i(I)| = 1$ gives $\Delta n_2(I) = 1$ and $\Delta n_1(I) = 0$ since 2 1-quint triangles are replaced with single 2-quint triangle.

   The interior of the T can give 1 2-quint triangle.

9. The number n_1 of 1-quint triangles can be estimated as follows.

   (a) Each pentagonal interior edge pair leading from $a(I, j)$ to $b(I, j)$ contributes 2 1-quint triangles for $\Delta(I) \neq \pm 1$, otherwise one obtains only 1 2-quint triangle. This would give maximum number of 6 1-quint triangles associated with the interior edges of 3 pentagons.

   (b) $P(I)$ pentagonal boundary edges contribute $2 \times (P(I) - 1)$ additional 1-quint triangles.

   (c) T contributes at most 4 1-quint triangles.

   (d) t can correspond 1-quint triangle and would do so if the interpretation of extended code is correct.

10. The construction also breaks the rotational symmetry since the decomposition of icosahedron to regions is like gauge fixing so that one can hope of obtaining only single representative in each

equivalence class of cycles and therefore less than 2^{10}. By the previous argument related to icosatetrahedral code, t and the triangle opposite to it cannot however correspond to amino-acids coded by 1 codon as one might guess first. Rather, t corresponds to "empty" amino-acid and to 1-quint triangle belonging to Z_2 orbit.

The number of cycles should be 2^{10}. One can try to estimate this number from the construction. Each $b_{I,j}$ can be chosen in 4 manners at the first step but at later steps some vertices of the neighboring pentagon might have been already visited and this reduces the available vertices by $n+1$ if n subsequent edges are visited. At each vertex $b_{I,j}$ one has 4 options for the choice of the boundary edges unless some boundary edges of pentagon (shared with other pentagons) have been already visited. It is also possible that the number of boundary edges vanishes. One can start from any vertex of triangle. This gives the upper bound of 2^4 choices giving $N < 2^{12}$ paths going through 4 pentagon-like regions. The condition that the path is closed, poses constraints on the edge path assignable to T but the number of choices is roughly 24. The condition that path goes through all vertices and that no edge is traversed twice must reduce this number to 2^{10}.

The numerical construction of Hamiltonian cycles should keep account about the number of vertices visited and this would reduce the number of candidates for $b(I,j)$ and for the choices of $P(I)$ for $I > 1$ as well as the number of edge paths associated with T.

3.4 Icosahedral Hamiltonian cycles numerically

A couple of months after writing the article I decided to look at the numerical problem of calculating the Hamiltonian cycles for icosahedron. Recall that the earlier source [A2] (`http://mathoverflow.net/questions/37788/why-are-there-1024-hamiltonian-cycles-on-an-icosahedron`) telling that there are 2^{10} different Hamiltonian cycles when orientation is taken into account and one edge is fixed: if orientation does not matter there re 2^9 cycles. If one does not fix one cycle one obtains 2560 cycles - not Hamiltonian paths as I had erratically concluded. The cycles were actually listed (`http://cs.smith.edu/~orourke/MathOverflow/hpaths.html`) and classified to five different basic classes according to their symmetries. Even better, examples of cycles with symmetries were illustrated.

Cycles can be divided to isomorphy classes within which cycles have same shape.

1. It is possible to perform a shift of the edges along the cycle. The shape of the cycle is not affected but cycle changes. Using music terms the key changes. There are 12 different keys.

2. Also the mirror image mapping i^{th} edge to $(13-i)^{th}$ edge is a symmetry which in the generic case produces a new cycle. This symmetry should be distinguished from the change of the internal orientation which does not affect the cycle.

3. Also the isometries of icosahedron leaving the fixed edge as such act as symmetries. Fixed edge belongs to a triangle and the reflection mapping the two other edges of the triangle to each other is this kind of symmetry. Therefore there are two reflection symmetries and the number of cycles of same shape in the generic case is expected to be $4 \times 12 = 48$. If some of the symmetries acts trivially or if some isometries of icosahedron act as its symmetries, the number of isomorphic cycles is reduced.

It is even possible to find illustrations of the symmetric cycles (`https://www.flickr.com/photos/edwynn/sets/72157625709580605/`) obtained using Brendan McKay's NAUTY software (`http://cs.anu.edu.au/~bdm/nauty/`)! From these illustrations (see Figs. 3.4, 3.4 and 3.4) one can by visual inspection deduce the numbers (n_0, n_1, n_2) charactering the cycle for classes involving symmetries. Also the basic chords can be deduced. If one trusts the condition $n_1 + 2 \times n_2 = 24$, it is enough to count the number n_2 triangles containing to path edges. I have also directly checked that n_1 comes out correctly.

There are following isomorphic collections.

1. 6 asymmetric collections containing the maximal number of 48 cycles each. In this case images are not given.

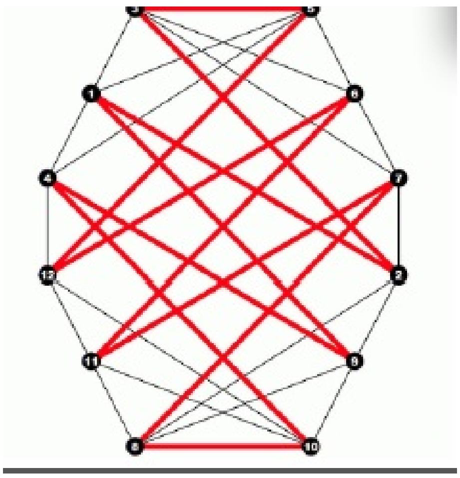

Figure 7: $((n_0, n_1, n_2) = (4, 8, 8)$ Hamiltonian cycle with 2 reflection symmetries acting in vertical and horizontal directions.

2. 3 collections with 2-fold rotation symmetry containing 48/2=24 cycles each. One has $(n_0, n_1, n_2) \in \{(0, 16, 4), (0, 16, 4), (4, 8, 8)\}$.
3. 5 collections with reflectional symmetry containing 48/2=24 cycles each. One has $(n_0, n_1, n_2) \in \{(2, 12, 6), (2, 12, 6), (4, 8, 8), (2, 12, 6), (2, 12, 6)\}$.
4. 2 collections with 2 reflectional symmetries containing 48/4=12 cycles each. One has $(n_0, n_1, n_2) \in \{(0, 16, 4), (4, 8, 8)\}$.
5. 1 collection with 6-fold rotational symmetry containing 48/6= 8 cycles. One has $(n_0, n_1, n_2) = (2, 12, 6)$.

There are therefore 5 different notions of harmony and they correspond to $n = \{6, 3, 5, 2, 1\}$ sub-harmonies. This gives altogether 6+3+5+2+1= 17 different notions of harmony.

What is remarkable that the original candidate $(3, 10, 7)$ for bio-harmony is not realized as a cycle possessing symmetries (it might be realized as one of the asymmetric cycles) but that there are at least three realizations for $(4, 8, 8)$, which is forced by the condition that bio-harmony corresponds to the extended genetic code! The three $(4, 8, 8)$ cycles are illustrated in Figs. 3.4, 3.4 and biorefl23.

4 Other ideas

The book of Merrick discusses also other ideas. The attempts to understand music in TGD framework relate to these ideas.

4.1 p-Adic length scale hypothesis and music

One of the key ideas is the reduction of the octave phenomenon to the p-adic length scale hypothesis predicting that octaves and half-octaves correspond to p-adic scalings allowed by the hypothesis $p \simeq 2^k$ for the preferred values of the p-adic primes, and yielding scaled variants of physical systems. This idea will not be discussed in the following: suffice it to say that Pythagorean scale coming as powers of $p = 3$ strongly suggests approximate 3-adicity.

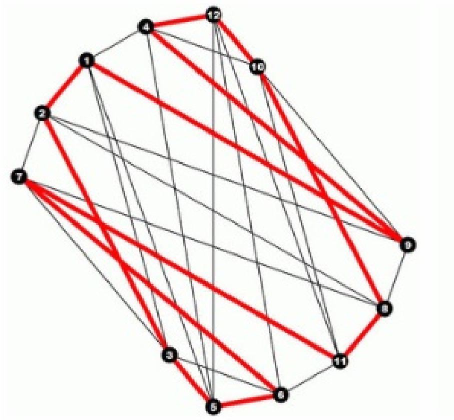

Figure 8: $((n_0, n_1, n_2) = (4, 8, 8)$ Hamiltonian cycle with 2-fold rotational symmetry acting as 6-quint rotation.

4.2 EEG and music

First of the key ideas relates to the idea that genetic code relates to the music scale.

1. Music metaphor is key element of TGD inspired view about biology and neuroscience. In particular, TGD based view about dark matter leads to the proposal that bio-photons are ordinary photons resulting as transformations of dark photons with large Planck constant $h_{eff} = nh$ to ordinary photons. The further hypothesis is that the energy spectrum of bio-photons is universal and contains visible photons and UV photons, which defined transition energies of biomolecules. This hypothesis follows if the value of h_{eff} assignable to a magnetic flux tube characterizes ion and is proportional to its mass number. The notion of gravitational Planck constant identified as $h_{gr} = GMm/v_0$, where v_0 is a velocity parameter assignable to the two-particle system can be identified in the case of elementary particles and ions with h_{eff} and predicts also the universality of bio-photon spectrum.

2. In this framework bio-photons would represent music as light inducing molecular transitions. Notes that is different energies of bio-photons would correspond to different magnetic field strengths at magnetic flux tubes as was proposed much earlier in the quantum model of hearing [K4]. Could the biochemical and physiological aspects involved with the generation of music experience be realized in terms of bio-photon emission induced by the listening of music?

4.3 Standing waves and music

Merrick consider the idea that standing waves are essential for music experience. Preferred extremals of Kähler action representing standing waves does not seem to be feasible. The known preferred extremals (with "massless extremals" (MEs) included) would represent superpositions of Fourier components with four-wave-vectors which are proportional to each other. Essentially pulse propagating in fixed direction. For more general extremals this direction can depend on position.

Although standing waves are not feasible, effects which would be explained in Maxwell's theory in terms of standing waves are possible in many-sheeted space-time. A particle in a region of Minkowski space containing several space-time sheets touches all space-time sheets having non-vanishing Minkowski space projection to this region and the forced experience by it is sum of the forces caused by them. This leads to an operational defines of gravitational and gauge fields of Einstein-Maxwell limit of TGD as sum of the deviations of the induced metric from Minkowski metric and sum of the components of the induced spinor connection defining classical gauge potentials in TGD framework.

Test particles can clearly experience the presence of standing waves. It is enough to take two massless extremals with opposite directions of three momentum but same energy with non-empty projections to

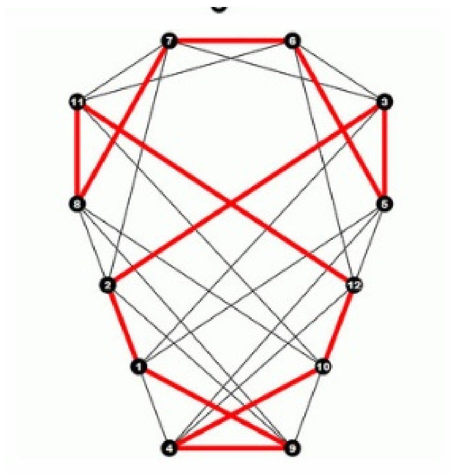

Figure 9: $((n_0, n_1, n_2) = (4, 8, 8)$ Hamiltonian cycle with 2-fold refection symmetry acting as horizontal reflection.

same M^4 region. Particle with experience standing wave oscillating with the frequency involved. The arrangements in which photons are taken to rest effectively could correspond to this kind of situations since if it is the motion of test particles which serves as a signature. Note however that there are also vacuum extremals for which the light velocity at the space-time surface corresponds to arbitrarily low velocity at the level of imbedding space.

4.4 Emotions and 4-D character of music experience

Music experience involves in an essential manner time unlike visual experience which is essentially 3-dimensional. Music experience affects also emotions very directly. For instance, we somehow know the key of the piece and expect that it ends to the basic note and chord. We somehow know also the scale used (say major or minor) by the emotional response stimulated by it. All this requires information about entire time evolution of the music piece. The recent neuroscience based models of memory do not help much in attempts to understand how this is possible. The reason is that in the ordinary materialistic view in which the state of the brain at fixed time should determine the contents of consciousness.

The general vision in Zero Energy Ontology and Quantum Classical Correspondence is that space-time surface provide classical physics correlates for quantum states and also quantum jumps: the failure of the strict determinism is essential for the latter. The space-time surfaces are restricted inside causal diamond (CD) and have space-like 3-surface as their ends: the interpretation is as counterparts for the initial and final states of physical events.

The replacement of states with events makes it possible to understand mysterious looking facts about living matter such as standardized temporal patterns - say those appearing during morphogenesis. The maxima of the vacuum function defined by the exponent of Kähler function in term identified as Kähler action for Euclidian space-time regions representing analogs for the lines of Feynman graph correspond to the most probably temporal patterns.

The basic aspect of emotions is positive/negative dichotomy. An attractive identification for the physical correlated of this aspect is whether the quantum jump generating the emotion increases or decreases the negentropy of the subsystem involved. For instance, pain would correspond to a reduction of the negentropy for the body part involved. In music experience negentropy could flow between different parts of the system involved and create also sensation with local negative coloring but with overall positive coloring (by NMP [K3]). The ability of temporal patterns of music to generate negentropy flows inside the system involved could explain its effectiveness in generating emotions.

Dissonances were used by composes like Bach to generate melancholic emotions which suggests that the dissonance represent local reduction of negentropy. Also vibrato has emotional content. Physically dissonance and vibrato are assignable to the interference of frequencies which are near to each other (http://en.wikipedia.org/wiki/Beat_(acoustics)). The basic formula is

$$cos(x) + cos(y) = cos((x + y)/2) \times cos((x - y)/2) \ .$$

Acknowledgements: I want to thank Tommi Ullgren for directing my attention to the book of Richard Merrick as well as for fascinating discussions about music.

REFERENCES

Mathematics

[A1] Icosahedral graph. Wolfram MathWorld. http://mathworld.wolfram.com/IcosahedralGraph.html.

[A2] Why are there 1024 Hamiltonian cycles on an icosahedron? http://mathoverflow.net/questions/37788/why-are-there-1024-hamiltonian-cycles-on-an-icosahedron.

[A3] B. Hopkins. Hamiltonian paths on Platonic graphs. *IJMMS. http://www.emis.de/journals/HOA/IJMMS/2004/29-321613.pdf*, 30:1613–1616, 2004.

Biology

[I1] Amino acid. http://en.wikipedia.org/wiki/Amino_acid.

[I2] Genetic code. http://en.wikipedia.org/wiki/Genetic_code.

Neuroscience and Consciousness

[J1] R. Merrick. *Interference: A Grand Scientific Musical Theory. http://interferencetheory.com/files/Interference.pdf*. 2009.

Books related to TGD

[K1] M. Pitkänen. DNA as Topological Quantum Computer. In *Genes and Memes*. Onlinebook. http://tgdtheory.fi/public_html/genememe/genememe.html#dnatqc, 2006.

[K2] M. Pitkänen. Genes and Memes. In *Genes and Memes*. Onlinebook. http://tgdtheory.fi/public_html/genememe/genememe.html#genememec, 2006.

[K3] M. Pitkänen. Negentropy Maximization Principle. In *TGD Inspired Theory of Consciousness*. Onlinebook. http://tgdtheory.fi/public_html/tgdconsc/tgdconsc.html#nmpc, 2006.

[K4] M. Pitkänen. Quantum Model for Hearing. In *TGD and EEG*. Onlinebook. http://tgdtheory.fi/public_html//tgdeeg/tgdeeg/tgdeeg.html#hearing, 2006.