Finite fields and TGD

November 6, 2022

Matti Pitkänen

Email: matpitka6@gmail.com.
http://tgdtheory.com/public_html/.
Recent postal address: Rinnekatu 2-4 A 8, 03620, Karkkila, Finland.

Contents

Abstract

TGD involves geometric and number theoretic physics as complementary views of physics. Almost all basic number fields: rationals and their algebraic extensions, p-adic number fields and their extensions, reals, complex number fields, quaternions, and octonions play a fundamental role in the number theoretical vision of TGD.

Even a hierarchy of infinite primes and corresponding number fields appears. At the first level of the hierarchy of infinite primes, the integer coefficients of a polynomial Q defining infinite prime have no common prime factors. $P = Q$ hypothesis states that the polynomial P defining space-time surface is identical with a polynomial Q defining infinite prime at the first level of hierarchy.

However, finite fields, which appear naturally as approximations of p-dic number fields, have not yet gained the expected preferred status as atoms of the number theoretic Universe. Also additional constraints on polynomials P are suggested by physical intuition.

Here the notions of prime polynomial and concept of infinite prime come to rescue. Prime polynomial P with prime order $n = p$ and integer coefficients smaller than p can be regarded as a polynomial in a finite field. The proposal is that all physically allowed polynomials are constructible as functional composites of prime polynomials satisfying $P = Q$ condition.

1 Introduction

This article represents some material related to two articles discussing number theoretical vision of TGD. The first article [L2] was about the fusion of geometric and number theoretic views of TGD to single coherent theory.

Second article [L1] was about my attempts to understand Langlands correspondence, which postulates a deep correspondence between number theory and geometry, and its relation to the geometric and number theoretic views of TGD. Both articles led to two unexpected new ideas and because of the potential importance of these ideas, I decided to write a separate article raising these ideas to table, as one might say.

1.1 Brief summary of the basic mathematical notions behind TGD

The theoretical framework behind TGD involves several different strands and the goal is to unify them to a single coherent whole. This challenge was discussed in [L2].

TGD involves number theoretic and geometric visions about physics and M^8-H duality, analogous to Langlands duality, is proposed to unify them. Also quantum classical correspondence (QCC) is a central aspect of TGD. One should understand both the M^8-H duality and QCC at the level of detail.

The following mathematical notions are expected to be of relevance for this goal.

1. Von Neumann algebras, call them M, in particular hyperfinite factors of type II_1 (HFFs), are in a central role. Both the geometric and number theoretic side, QCC could mathematically correspond to the relationship between M and its commutant M'.

   For instance, symplectic transformations leave induced Kähler form invariant and various fluxes of Kähler form are symplectic invariants and correspond to classical physics commuting with quantum physics coded by the super symplectic algebra (SSA). On the number theoretic side, the Galois invariants assignable to the polynomials determining space-time surfaces are analogous classical invariants.

2. The generalization of ordinary arithmetics to quantum arithmetics obtained by replacing $+$ and $\times$ with $\oplus$ and $\otimes$ allows us to replace the notions of finite and p-adic number fields with their quantum variants. The same applies to various algebras.

3. Number theoretic vision leads to adelic physics involving a fusion of various p-adic physics and real physics and to hierarchies of extensions of rationals involving hierarchies of Galois groups involving inclusions of normal subgroups. The notion of adele can be generalized by replacing various p-adic number fields with the p-adic representations of various algebras.

4. The physical interpretation of the notion of infinite prime has remained elusive although a formal interpretation in terms of a repeated quantization of a supersymmetric arithmetic QFT is highly suggestive. One can also generalize infinite primes to their quantum variants. The proposal is that the hierarchy of infinite primes generalizes the notion of adele.

   Second proposal, discussed already in [L2] and to be discussed separately in this article, was that the polynomial Q defining infinite prime at the first level of the hierarchy are identical to the polynomial P defining 4-surface in M^8 and by M^8-H correspondence space-time surface in $H=M^4\times CP_2$. This would realize quantum classical correspondence at very deep level.

The formulation of physics as Kähler geometry of the "world of classical worlds" (WCW) involves f 3 kinds of algebras A; supersymplectic isometries SSA acting on $\delta M^4_+\times CP_2$, affine algebras Aff acting on light-like partonic orbits, and isometries I of light-cone boundary δM^4_+, allowing hierarchies A_n.

The braided Galois group algebras at the number theory side and algebras $\{A_n\}$ at the geometric side define excellent candidates for inclusion hierarchies of HFFs. M^8-H duality suggests that n corresponds to the degree $n$of the polynomial P defining space-time surface and that the n roots of P correspond to n braid strands at H side. Braided Galois group would act in A_n and hierarchies of Galois groups would induce hierarchies of inclusions of HFFs. The ramified primes of P would correspond to physically preferred p-adic primes in the adelic structure formed by p-adic variants of A_n with $+$ and $\times$ replaced with $\oplus$ and $\otimes$.

1.2 Langlands correspondence and TGD

In the article [L1], the TGD counterpart of Langlands program was discussed and this led as a side product to a realization how finite fields could serve as basic building blocks of the number theoretic vision of TGD.

1. Concerning the concretization of the basic ideas of Langlands program in TGD, the basic principle would be quantum classical correspondence (QCC), which is formulated as a correspondence between the quantum states in the "world of classical worlds" (WCW) characterized by analogs of partition functions as modular forms and classical representations realized as space-time surfaces. L-function as a counter part of the partition function would define as its roots space-time surfaces and these in turn would define via Galois group representation partition function. QCC would define a kind of closed loop giving rise to a hierarchy.

2. If Riemann hypothesis (RH) is true and the roots of L-functions are algebraic numbers, L-functions are in many aspects like rational polynomials and motivate the idea that, besides rationals polynomials, also L-functions could define space-time surfaces as kinds of higher level classical representations of physics.

3. One concretization of Langlands program would be the extension of the representations of the Galois group to the polynomials P to the representations of reductive groups appearing naturally in the TGD framework. Elementary particle vacuum functionals are defined as modular invariant forms of Teichmüller parameters. Multiple residue integral is proposed as a manner to obtain L-functions defining space-time surfaces.

4. One challenge is to construct Riemann zeta and the associated ξ function and the Hadamard product leads to a proposal for the Taylor coefficients c_k of $\xi(s)$ as a function of $s(s-1)$. One would have $c_k = \sum_{i,j} c_{k,ij} e^{i/k} e^{\sqrt{-1}2\pi j/n}$, $c_{k,ij} \in \{0, \pm 1\}$. $e^{1/k}$ is the hyperbolic analogy for a root of unity and defines a finite-D transcendental extension of p-adic numbers and together with n :th roots of unity powers of $e^{1/k}$ define a discrete tessellation of the hyperbolic space H^2.

   This construction led to the question whether also finite fields could play a fundamental role in the number theoretic vision. Prime polynomial with prime order $n = p$ and integer coefficients smaller than $n = p$ can be regarded as a polynomial in a finite field. If it satisfies the condition that the integer coefficients have no common prime factors, it defines an infinite prime. The proposal is that all physically allowed polynomials are constructible as functional composites of these.

One can end up to the idea that prime polynomials and finite fields could be fundamental in TGD also by a different route.

1. A highly interesting feedback to the number theoretic vision emerges. The rational polynomials P defining space-time surfaces are characterized by ramified primes. Without further conditions, they do not correlate at all with the degree n of P as the physical intuition suggests.

2. In [L2] it was proposed that P can be identified as the polynomial Q defining an infinite prime [K3]: this implies that the coefficients of the integer polynomial P (to which any rational polynomial can be scaled) do not have common prime factors.

3. An additional condition could be that the coefficients of P are smaller than the degree n of P. For $n = p$, P could as such be regarded as a polynomial in a finite field. This proposal is too strong to be true generally but could hold true for so-called prime polynomials of prime order having no functional decomposition to polynomials of lower degree [A1, A2]. The proposal is that all physically allowed polynomials are constructible as functional composites of these. Also finite fields would become fundamental in the TGD framework.

Because of the potential importance of this idea, which emerged while writing article about my attempts to understand Langlands correpondence and its relation to TGD, I decided to write a separate article about the role of finite fields in the TGD based world order.

2 Infinite primes as a basic mathematical building block

Infinite primes [K3, K1, K2] are one of the key ideas of TGD. Their precise physical interpretation and the role in the mathematical structure of TGD has however remained unclear.

3 new ideas are be discussed. Infinite primes could define a generalization of the notion of adele; quantum arithmetics could replace $+$ and $\times$ with $\oplus$ and $\otimes$ and ordinary primes with p-adic representations of say HFFs; the polynomial Q defining an infinite prime could be identified with the polynomial P defining the space-time surface: $P = Q$.

2.1 Construction of infinite primes

Consider first the construction of infinite primes [K3].

1. At the lowest level of hierachy, infinite primes (in real sense, p-adically they have unit norm) can be defined by polynomials of the product X of all primes as an analog of Dirac vacuum.

   The decomposition of the simplest infinite primes at the lowest level are of form $aX + b$, where the terms have no common prime divisors. More concretely $a = m_1/n_F$ $b = m_0 n_F$, where n_F is square free integer analogous and the integer m_1 and n_F have no common prime divisors divisors. The divisors of m_2 are divisors of n_F and m_i has interpretation as n-boson state. Power p^k corresponds to k-boson state with momenta p. $n_F = \prod p_i$ has interpretation as many-fermion state satisfying Fermi-Dirac statistics.

   The decomposition of lowest level infinite primes to infinite and finite part has a physical analogy as kicking of fermions from Dirac sea to form the finite part of infinite prime. These states have interpretation as analogs of free states of supersymmetric arithmetic quantum field theory (QFT) There is a temptation to interpret the sum $X/n_F + n_F$ as an analog of quantum superposition. Fermion number is well-defined if one assigns the number of factors of n_F to both n_F and X/n_F.

2. More general infinite primes correspond to polynomials $Q(X) = \sum_n q_n X^n$ required to define infinite integers which are not divisible by finite primes. Each summand $q_n X^n$ must be a infinite integer. This requires that q_n is given by $q_n = m_{B,n}/\prod_{i_1}^{n} n_{F,i}$ of square free integers $n_{F,i}$ having no common divisors.

   The coefficients $m_{B,n}$ representing bosonic states have no common primes with $\prod n_{F,i}$ and there exists no prime dividing all coefficients $m_{B,n}$: there is no boson with momentum p present in all states in the sum.

   These states have a formal interpretation as bound states of arithmetic supersymmetric QFT. The degree k of Q determines the number of particles in the bound states.

   The products of infinite primes at given level are infinite primes with respect to the primes at the lower levels but infinite integers at their own level. Sums of infinite primes are not in general infinite primes. For instance the sum and difference of $X/n_F + n_F$ and $X/n_F - n_F$ are not infinite primes.

3. At the next step one can form the product of all finite primes and infinite primes constructed in this manner and repeat the process as an analog to second quantization. This procedure can be repeated indefinitely. This repeated quantization a hierarchy of infinite primes, which could correspond to the hierarchy of space-time sheets.

   At the n:th hierarchy level the polynomials are polynomials of n variables X_i. A possible interpretation would be that one has families of infinite primes at the first level labelled by n_1 parameters. If the polynomials $P(x)$ at the first level define space-time surfaces, the interpretation at the level of WCW could be that one has an $n-1$-D surface in WCW parametrized by $n-1$ parameters with rational values and defining a kind of sub-WCW. The WCW spinor fields would be restricted to this surface of WCW.

The Dirac vacuum X brings in mind adele, which is roughly a product of p-adic number fields. The primes of infinite prime could be interpreted as labels for p-adic number fields. Even more generally, they could serve as labels for p-adic representations of various algebras and one could even consider replacing the arithmetic operations with $\oplus$ and $\otimes$ to get the quantum variants of various number fields and of adeles.

The quantum counterparts of nfinite primes at the lowest and also at the higher levels of hierarchy could be seen as a generalization of adeles to quantum adeles.

2.2 Questions about infinite primes

One can ask several questions about infinite primes.

1. Could $\oplus$ and $\otimes$ replace $+$ and $-$ also for infinite primes. This would allow us to interpret the primes p as labels for algebras realized p-adically. This would give rise to quantal counterparts of infinite primes.

2. What could $+ \to \oplus$ for infinite primes mean physically? Could it make sense in adelic context? Infinite part has finite p-adic norms. The interpretation as direct sum conforms with the fermionic interpretation if the product of all finite primes is interpreted as Dirac sea. In this case, the finite and infinite parts of infinite prime would have the same fermion number.

3. Could adelization relate to the notion of infinite primes? Could one generalize quantum adeles based on $\oplus$ and $\otimes$ so that they would have parts with various degrees of infinity?

2.3 $P = Q$ hypothesis

One cannot avoid the idea that that polynomial, call it $Q(X)$, defining an infinite prime at the first level of the hierarchy, is nothing but the polynomial P defining a 4-surface in M^4 and therefore also a space-time surface. $P = Q$ would be a condition analogous to the variational principle defining preferred extremals (PEs) at the level of H.

There is however an objection.

1. $P = Q$ gives very powerful constraints on Q since it must define an infinite integer. The prime polynomials P are expected to be highly non-unique and an entire class of polynomials of fixed degree characterized by the Galois group as an invariant is in question. The same applies to polynomials Q as is easy to see: the only condition is that powers of $a_k X^k$ defining infinite integers have no common prime factors.

2. It seems that a composite polynomial $P_n \circ ... \circ P_1$ satisfying $P_i = Q_i$ cannot define an infinite prime or even infinite integer. Even infinite integer property requires very special conditions.

3. There is however no need to assume $P_i = Q_i$ conditions. It is enough to require that there exists a composite $P_n \circ ... \circ P_1$ of prime polynomials satisfying $P_n \circ ... \circ P_1 = Q$ defining an infinite prime.

   The physical interpretation would be that the interaction spoils the infinite prime property of the composites and they become analogs of off-mass-shell particles. Exactly this occurs for bound many-particle states of particles represented by P_i represented composite polynomials $P_1 \circ ... P_n$. The roots of the composite polynomials are indeed affected for the composite. Note that also products of Q_i are infinite primes and the interpretation is as a free many-particle state formed by bound states Q_i.

There is also a second objection against $P = Q$ property.

1. The proposed physical interpretation is that the ramified primes associated with $P = Q$ correspond to the p-adic primes characterizing particles. This would mean that the ramimied primes appearing in the infinite primes at the first level of the hierarchy should be physically special.

2. The first naive guess is that for the simplest infinite primes $Q(X) = (m_1/n_F)X + m_2n_F$ at the first level, the finite part m_2n_F has an identification as the discriminant D of the polynomial $P(X)$ defining the space-time surface. This guess has no obvious generalization to higher degree polynomials $Q(X)$ and the following argument shows that it does not make sense.

   Since Q is a rational polynomial of degree 1 there is only a single rational root and discriminant defined by the differences of distinct roots is ill-defined that $Q = P$ condition would not allow the simplest infinite primes.

   Therefore one must give either of these conjectures and since $P = Q$ conjecture dictates the algebraic structure of the quantum theory for a given space-time surface, it is much more attractive.

The following argument gives $P = Q$. One can assign to polynomial P invariants as symmetric functions of the roots. They are invariants under permutation group S_n of roots containing Galois group and therefore also Galois invariants (for polynomials of second order correspond to sum and product of roots appearing as coefficients of the polynomial in the representation $x^2+bx+cx$). The polynomial Q having as coefficients these invariants is the original polynomial. This interpretation gives $P = Q$.

3 How also finite fields could define fundamental number fields in Quantum TGD?

One can represent two objections against the number theoretic vision.

1. The first problem is related to the physical interpretation of the number theoretic vision. The ramified primes p_{ram} dividing the discriminant of the rational polynomial P have a physical interpretation as p-adic primes defining p-adic length- and mass scales.

   The problem is that without further assumptions they do not correlate at all with the degree n of P. However, physical intuition suggests that they should depend on the degree of P so that a small degree n implying a low algebraic complexity should correspond to small ramified primes. This is achieved if the coefficients of P are smaller than n and thus involve only prime factors $p < n$.

2. All number fields except finite fields, that is rationals and their extension, p-adic numbers and their extensions, reals, complex numbers, quaternions, and octonsions appear at the fundamental level in TGD. Could there be a manner to make also finite fields a natural part of TGD?

These problems raise the question of whether one could pose additional conditions to the polynomials P of degree n defining 4-surfaces in M^8 with roots defining mass shells in $M^4 \subset M^8$ (complexification assumed) mapped by $M^8 - H$ duality to space-time surfaces in H.

3.1 $P = Q$ condition

One such condition was proposed in [L2]. The proposal is that infinite primes forming a hierarchy are central for quantum TGD. It is proposed that the notion of infinite prime generalizes to that of the notion of adele.

1. Infinite primes at the lowest level of the hierarchy correspond to polynomials of single variable x replaced with the product $X = \prod_p p$ of all finite primes. The coefficients of the polynomial do not have common prime divisors. At higher levels, one has polynomials of several variables satisfying analogous conditions.

2. The notion of infinite prime generalizes and one can replace the argument x with Hilbert space,group representation, or algebra and sum and product of ordinary arithmetics with direct sum $\oplus$ and tensor product $\otimes$.

3. The proposal is $P = Q$: at the lowest level of the hierarchy, the polynomial $P(x)$ defining a space-time surface corresponds to an infinite prime determined by a polynomial $Q(X)$. This would be one realization of quantum classical correspondence. This gives strong constraints to the space-time surface and one might speak of the analog of preferred extremal (PE) at the level of M^8 but does not yet give any special role for the finite fields.

4. The infinite primes at the higher level of the hierarchies correspond to polynomials $Q(x_1, x_2, ..., x_k)$ of several variables. How to assign a polynomial of a single argument and thus a 4-surface to Q? One possibility is that one does as in the case of multiple poly-zeta and performs a multiple residue integral around the pole at infinity and obtains a finite result. The remaining polynomial would define the space-time surface.

3.2 Additional conditions

The speculations related to the p-adicization of the ξ function associated with the Riemann zeta discussed in [L1] inspired the following questions.

1. Option I: Rational polynomial is apart from scaling a polynomial with integer coefficients having the same roots. Could it make sense to assume that the coefficients of the $P(x) = Q(x)$ of degree n are integers divisible only by primes $p < n$?

2. Option II: A stronger condition would be that the integer coefficients of $P = Q$ are smaller than n. This implies that they are divisible by primes $p < n$, which cannot however appear as common factors of the coefficients. One could say that the corresponding space-time sheet effectively lives in the ring Z_n instead of integers. For prime value $n = p$ space-time sheet would effectively "live" the finite field F_p and finite fields would gain a fundamental status in the structure of TGD.

   Should one allow both signs for the coefficients as the interpretation as rationals would suggest? In this case, finite field interpretation would mean the replacement of -1 with $p-1$.

   The construction of the proposed polynomials is very simple. Only integers, having as their factors primes $p < n$, are possible as coefficients $p_n$of P and p_n have no common prime divisors. One can imagine n boxes to, which one puts integers $m < n$ decomposing into prime factors $p < n$. Also $m \in \{0, \pm 1$ are allowed. Single box can contain several primes but the same prime can appear only in a single box. This is like having Bose-Einstein condensates of bosons labelled by primes, each localized to a single box, which can contain several Bose-Einstein condensates.

   The number of boxes containing primes cannot be larger than the number $N(p, n)$ of primes $p < n$. If m different integers $m > 1$ are involved, the number of possible distributions of boxes containing these integers is $B(n, k)$. There is also degeneracy related to the distribution of 1:s and 0:s among remaining boxes.

3. Option III: A still stronger, perhaps too strong, condition would be that only the prime factors of n appear as factors of the coefficients of $P = Q$. For integers n with a small number of prime divisors it is easy to find the possible coefficients. For instance, for $n = p$ all coefficients are equal to 1 or 0!

   For $n = p_1p_2$, two of the coefficients can be equal to power of p_1 or p_2 if smaller than n and remaining coefficients equal to 1 or 0. For instance, $n = p_1p_2$ for $p_1 = M_{127} = 2^{127} - 1$ and $p_2 = 2$, one coefficient could be M_{127}, second coefficient power of 2 smaller than 2^{127} and the remaining coefficients would be equal to 1 or 0.

Option II would solve the two problems whereas Option III is un-necessarily strong.

1. For $n = p$, P would make sense in a finite field F_p if the second condition is true. Finite fields, which have been missing from the hierarchy of numbers fields, would find a natural place in TGD if this condition holds true!

2. The number of polynomial coefficients is n, whereas the number of primes smaller than n behaves as $n/log(n)$. By infinite prime property, the coefficients would not contain common primes $p < n$. Very few polynomials could define space-time surfaces.

3.3 How does Option II relate to prime polynomials?

One can invent an objection against Option II. One of the basic conjectures of the number theoretic vision has been that functional composition of polynomials $P = P_2 \circ P_1$ of degrees m and n giving more complex polynomials is possible. This would give rise to evolutionary hierarchies and could also correspond to the inclusion hierarchies for hyperfinite factors of type II_1. The additional assumption has been that the polynomials vanish at $x = 0$ that $P_0 = 0$.

In the $n = 3$ case, the composite $P_1 \circ P_1$ for $P_1 = x + 2x^2$, is $x + 4x^2 + 8x^3 + 8x^4$ and fails to satisfy the conditions.

Could the proposed conditions hold true for so-called prime polynomials, which are analogous to infinite primes? Prime polynomials are discussed in [L2].

1. Polynomials can be factorized into composites of prime polynomials [A1, A2] (`https://cutt.ly/HXAKDzT` and `https://cutt.ly/5XAKCe2`). A polynomial, which does not have a functional composition to lower degree polynomials, is called a prime polynomial. It is not possible to assign to prime polynomials prime degrees except in special cases. Simple Galois groups with no normal subgroups must correspond to prime polynomials.

2. For a non-prime polynomial, the number N of the factors P_i, their degrees n_i are fixed and only their order can vary so that n_i and $n = \prod n_i$ is an invariant of a prime polynomial and of simple Galois group [A1, A2]. Note that this composition need not exist for monic polynomials even if the Galois group is not simple so that polynomial primes in the monic sense need not correspond to simple Galois groups.

How does Option II relate to prime polynomials?

1. The degree of a composite of polynomials with orders m and n is mn so that a polynomial with prime degree p does not allow expression as a composite of polynomials of lower orders so that any polynomials with prime order is a prime polynomial. Polynomials of order m can in principle be functional composites of prime polynomials with orders, which are prime factors of m.

   Obviously, all prime polynomials cannot satisfy Option II. However, those satisfying Option II could be prime polynomials.

2. There are also non-prime polynomials satisfying Option II. $P_1 = x^m$ and $P_2 = x^n$ satisfy Option II as also the composite $P = x^{mn}$, which is however not a prime polynomial. The composite of $P_1 = x^2$ and $P_2 = 1+x^m$ gives $P = 1+2x^m+x^{2m}$, which satisfies Option II but is not prime. By the symmetry $B(n,k) = B(n,n-k)$ of binomial coefficients the composite of $P_1 = x^m$, $m > 2$, and $P_2 = 1 + x^m$ does not satisfy the conditions.

3. Quite generally, polynomials P satisfying Option II and having degree n, which is not prime, can decompose to prime polynomials and probably do so. There the polynomial primeness and Option II do not seem to have a simple relationship.

These observations suggest the tightening of the Option II to the following condition.

All physically allowed polynomials P are functional composites of the prime polynomials of prime degree satisfying Option II. In a rather precise sense, finite fields would serve as basic building blocks of the Universe.

Note that the polynomials, which have an interpretation in terms of a finite field F_p have degree $p-1$ and would therefore have a decomposition to a functional composite of prime polynomials satisfying Option II. On the other hand, polynomials with degree $p+1$ could reduce to prime polynomial of degree p.

p-Adic length scale hypothesis states that primes near powers of two and possibly also primes near powers of other small primes are favoured as p-adic primes identified as ramified primes. Mersenne primes $M_k = 2^k - 1$ are maximally near to a power of 2 and $n = 2^k$ would correspond to $p+1$. The polynomial $P = px^2 - 1$ has as its roots $x_\pm = \pm 1/\sqrt{p}$. The roots are not affected much if one adds to P large enough powers of x, say x^p, to get prime polynomial order p satisfying Option II since for the roots one has $x_{pm}^p \simeq \pm p^{-1/2p}$.

3.4 Examples of Option II

The following examples illustrate the conditions for Option II.

1. For instance, for $M_{127} = 2^{127} - 1$ assigned with electron by p-adic mass calculations one has $n/log(n) \simeq M_{127}/log(2)127 \simeq M_{127}/88$ so that only about 12 percent of coefficients of P could differ from 0 or 1.

2. For small values of n it is easy to construct the possible polynomials P.

   (a) For $n = p = 2$ one obtains only the coefficients $(p_0, p_1) \subset \{\pm 1, 0\}, \{0, \pm 1\}, \{\pm 1, \pm 1\}$ corresponding to $P(x) \in \{\pm 1, \pm x, \pm 1 \pm x\}$.

   (b) For $n = p = 3$, one of the coefficients is $p = 2$ and the remaining coefficients are equal to 1 or 0. The coefficients are $(p_0, p_1, p_2) \subset \{\pm 2, x, y\}, \{x, \pm 2, y\}, \{x, y, \pm 2\}$ with $x, y \in \{0, 1, -1\}$ and (p_0, p_1, p_2) with $p_i \in \{0, 1, -1\}$.
   A little calculation shows that extensions of rationals containing i, $\sqrt{2}$, $i\sqrt{2}$, $\sqrt{3}$, $i\sqrt{3}$, $\sqrt{5}$ (from $P = x^2 + x - 1$ defining Golden Mean), and $i\sqrt{7}$ are obtained.

   (c) Roots of small primes appear in the Weyl groups, which are reflection groups associated with Dynkin diagrams characterizing Lie groups at Lie algebra level. The finite discrete subgroups of the rotation group $SU(2)$ characterized extensions of hyper-finite factors of type II_1 and roots of small primes appear in the matrix elements of these groups. Could the proposed polynomials give in a natural way rise to the extensions of rationals appearing in these two cases?

The above considerations inspire further questions. Could one also allow polynomials P having coefficients in an algebraic extension of rationals? Does this bring in anything new? Could one have coefficients in an extension containing e or even root of e as perhaps the only transcendental extension defining a finite extension of p-adic numbers? The roots would be generalizations of algebraic numbers involving e and could make sense p-adically via Taylor expansion.

REFERENCES

Mathematics

[A1] Beardon AF. Composition factors of polynomials. *Complex Variables, Theory and Application: An International Journal*, 43(3-4):225–239, 2001. Available at: `https://doi.org/10.1080/17476930108815314`.

[A2] Ritt JF. Prime and Composite Polynomials. *Transactions of the American Mathematical Society*, 23(1):51–66, 1922. Available at: `https://doi.org/10.2307/1988911`.

Books related to TGD

[K1] Pitkänen M. Infinite Primes and Consciousness. In *Mathematical Aspect of Consciousness.* Available at: `https:/tgdtheory.fi/pdfpool/infpc.pdf`, 2006.

[K2] Pitkänen M. Motives and Infinite Primes. In *TGD as a Generalized Number Theory.* Available at: `https:/tgdtheory.fi/pdfpool/infmotives.pdf`, 2011.

[K3] Pitkänen M. TGD as a Generalized Number Theory: Infinite Primes. In *TGD as a Generalized Number Theory: Part I.* Available at: `https:/tgdtheory.fi/pdfpool/visionc.pdf`, 2019.

Articles about TGD

[L1] Pitkänen M. Some New Ideas Related to Langlands Program *viz.* TGD. https://tgdtheory.fi/public_html/articles/Langlands2022.pdf., 2022.

[L2] Pitkänen M. Trying to fuse the basic mathematical ideas of quantum TGD to a single coherent whole. https://tgdtheory.fi/public_html/articles/fusionTGD.pdf., 2022.