

This Time – What a Strange Turn of Events! by Philip E. Gibbs

Abstract

In relativity time is bound to space by the symmetries of spacetime. In the general theory the symmetry is covariance under diffeomorphisms but in string theory this extends to the full permutation group acting on spacetime events. This huge symmetry has profound implications for the nature of time, causality and the way we see our place in the universe.

The Relativity of Time

This year marked the hundredth anniversary of a talk delivered by Hermann Minkowski in Cologne. “Henceforth space by itself, and time by itself, are doomed to fade away into mere shadows, and only a kind of union of the two will preserve an independent reality.” Since that day it has not been possible to talk about the nature of time without also talking about the nature of space. A **symmetry** binds them into one entity called **spacetime**. In physics symmetry is linked to conservation laws by Noether’s theorem. From a given symmetry in the dynamics, Noether showed how to construct a conservation law. Even as separate entities, time and space are subject to symmetries. The underlying laws of physics do not change in time and they are the same everywhere in space. Without such Copernican principles science as we know it today would be impossible. We would need different laws of physics for different places and different times. Yet these principles of invariance are simple rules of symmetry in just the same way as the symmetry of a flower tells us that its form does not change as it is rotated. Using these symmetries of space and time and the formula of Noether we can show that the total amounts of momentum and energy are unchanged by the passage of time.

Information and the arrow of time

Minkowski used the symmetry in the Lorentz transformation to bring together space and time making them merely different dimensions of spacetime. Yet time is somehow different in our mind. This difference is characterised by an **arrow of time** that defies the symmetry. In our conscious experience our past is clear and fixed but our future is uncertain. From the laws of thermodynamics we learn that this difference is due to **entropy** which always increases as time passes. Entropy is a measure of **information** and by the rules of quantum mechanics information is conserved. There is a paradox, but information can be as clear to us as the letters on this page, or as hidden and disordered as the states of the molecules in the air around it. As time passes, the disorder increases and entropy measures this change.

Time can distinguish itself from space in this way because the spacetime metric has a **Lorentz signature** that assigns a different sign in the time dimension versus the three space directions. Thus in locally flat Minkowski spacetime distances are measured by the invariant quantity

$$ds^2 = dx^2 + dy^2 + dz^2 - c^2 dt^2$$

Part of the mystery of time is to understand where this signature comes from. Why three plus signs for space and only one minus sign for time? Even with this separation of dimensions there should remain symmetry under time reversal $t \rightarrow -t$, but the arrow of time breaks this symmetry. What is the origin of this arrow? From what bow did it take flight?

Relational Physics

A few years after Minkowski's talk, Einstein gave us the **equivalence principle**. Einstein was steeped in the philosophies of Ernst Mach and Immanuel Kant whose inspirations can be traced back centuries to the relational monadology of Leibniz. Einstein's great triumph was to develop the theory **general relativity** from Newtonian gravity and electrodynamics with little more guidance than these few principles and the need for mathematical beauty and consistency. The equivalence principle became a principle of symmetry going far beyond the symmetry that Minkowski had used to unite space and time. Noether's theorem still applies. Energy and Momentum are still strictly conserved. If you follow through Noether's theorem in general relativity staying faithful to the relativity principle you can show that it works perfectly, as I did in my 1997 paper "Covariant Energy-Momentum Conservation In General Relativity" ^[1]

Yet if general relativity was Einstein's great success, it was also a disappointment to him. He had wanted to remove space and time altogether and produce a theory based purely on relations between material objects as demanded by Mach. Instead he created a theory of dynamical geometry where spacetime itself took on a material existence. Still - it worked for classical gravity, and that's what counts.

Since the advent of quantum field theory we know that spacetime is filled with the virtual interaction of particles even when filled only by vacuum. This allows us to think again about relational models knowing that even the dynamics of empty spacetime may arise from relations between virtual particles. It may be 17th century philosophy but relational physics is completely right for the 21st century pursuit of a quantum theory of gravity.

The origin of Time

General Relativity also provides us with an answer to our question about the origin of time's arrow. It is a controversial answer because it depends on cosmology, yet where else could the answer lie? We can trace the influence of time's arrow on Earth back to the Sun as the source of almost all our energy. From the Sun we go back further beyond its formation and back further still to the big bang itself. Entropy has been increasing steadily since the big bang, but only because it started so low.

So we know that the source of the direction of time is the big bang - the cosmological **singularity** of spacetime predicted using the theory of general relativity - but we still don't know why. The physics of singularities is tied to the laws of **quantum gravity** that remain poorly understood. If low entropy is a quantum property of singularities, does that mean entropy will decrease if you fall into a black hole? Will time's arrow reverse as you approach its future singularity? For a while some physicists including Stephen Hawking considered that this might be true, but only if the future singularity itself was as big as the initial one. This was the case in a model of the universe called **Gold's Universe** where a closed spacetime expanded from the big bang until it reached a maximum size. From there it would collapse until it hit another singularity at the big crunch.

Hawking entertained the possibility that time would run backwards in the closing half of the universe's history. Even black holes would go into reverse becoming white holes. Finally he was persuaded that this scenario could not be correct and he declared it his greatest mistake, but the reasons for his capitulation are based on very speculative quantum cosmological models of the big bang and are not convincing. At the time Gold's Universe was a believable cosmology consistent

with what we knew about the state of the universe. Today we know that there is a force of dark energy causing the universe's expansion to accelerate and no big crunch is likely. There is no longer any reason to question the idea that the arrow of time is just a local effect driven by the big bang singularity.

The Big Bang as a White Hole

A few years ago I was a keen contributor of articles for the Usenet Physics FAQ and one of the frequent questions needing answered was "Is the universe a black hole?" The conventional cosmologists answer was "No!" The big bang singularity is in the past but the singularity of a black hole lies in the future light-cone so there is a clear qualitative difference. OK, but does that mean that the big bang could be a **white hole**? A white hole is the time reversal of a black hole and is an equally valid solution of the equations of general relativity. Again the conventional answer is "No!" Our favoured homogenous models of the big bang are still qualitatively different from a white hole, and in any case a white hole can never be created for the same reasons that a black hole can never be destroyed. There is a problem with this dismissal. We don't know that the universe is homogenous beyond the observable event horizon. If you drop this assumption known as the **Cosmological Principle**, then the universe could indeed be a white hole. A solution of general relativity known as the **Lemaître-Tolman model** can describe either the in-fall of matter into a black hole or - when time reversed - a **non-homogeneous** model of the big bang. As I demonstrated in my paper "A White Hole Model of the Big Bang" ^[2] there is no observation from within the observable universe that could distinguish it from a very large white hole.

Our natural intuition to discard such models of the universe is based on a misguided faith in **temporal causality**. We don't want to believe that black holes could run backwards as white holes but physics tells us otherwise. Research by Lawrence Schulman^[3] has shown that there is no conflict between zones where time runs in opposite directions. On the largest scales of our universe, time reversal symmetry may be restored, with local patches where time's arrow runs in one direction or the other influenced by the presence of the singularities in black or white holes many billions of light years across.

Event-Symmetric Spacetime

The argument so far: Time is linked inseparably to space through symmetry. This is embodied in the theory of general relativity where the symmetry is invariance under smooth changes of co-ordinates, also known as **diffeomorphism invariance**. It is partly relational but a more complete theory of *quantum* gravity should go further. Spacetime and its dynamics should be an **emergent** feature of the relations between particles. Despite the symmetry, time has a different character from space because it is linked to increasing disorder as information changes. However the arrow of time is a cosmological influence left over from the big bang, not a fundamental property of the laws of physics. We should not build temporal causality into our fundamental theory.

Matter and Feynman Diagrams

How can this be achieved? How can we remove time and causality yet keep their symmetry? In quantum field theory matter can be described by particles that interact via **Feynman diagrams**. The vertices of the diagram are labelled with events in spacetime, and we must integrate over all the possible co-ordinates they can take. This interaction process is described by creation and annihilation operators for each particle state i and each spacetime event x . The set of these

operators $\{a_i(x), a_i^*(x)\}$ generate an algebra to which the **Hamiltonian** belongs. So let's drop the notions of space and time keeping just a set of operators $\{a_i, a_i^*\}$ labelled only by unknown states. The Feynman diagrams remain but now they are just graphs with vertices representing interactions and a sum over states in place of the integrals over spacetime events.

What symmetry is retained in such a model? Only the **permutation symmetry** related to the **indistinguishability** of particles remains. It is characterized by the commutation relations between the operators $[a_i, a_j^*] = \delta_{ij}$. A change of co-ordinates is also a permutation of space-time events, but only permutations of events described by smooth functions between co-ordinates are allowed in general relativity. In a purely relational model the events should become the discrete nodes in the Feynman diagrams and the symmetry should become the full permutation group on these events. We can assign dynamical rules to the Feynman diagrams but the rules must respect the permutation symmetry. Our goal is to assign rules in such a way that spacetime is an emergent property of the dynamics and the symmetry of general co-ordinate transformations is a remnant of the permutation symmetry of the particle operators.

Random Graphs

Based on this idea, the simplest models of relational physics are **random graphs** representing matter by Feynman diagrams. The nodes of the graphs are events of space time and the edges of the graphs represent the passage of particles. In a **Lagrangian** approach the dynamics of the graph are determined by an **action** $S(G)$ which is a function of the graph configuration G depending on parameters such as the total number of links or the valence of the nodes. We study the physical properties of the system using its partition function.

$$Z = \sum_G e^{iS(G)}$$

We are free to choose a form for the action except that it must be invariant under the symmetry group acting as permutations on the graph nodes. This symmetry embodies both the indistinguishability of particles and the diffeomorphism invariance of spacetime. The beauty of the random graph model is that the dimension and signature of spacetime cannot be built into the action by hand. The event symmetry forbids it. These must be emergent consequences of the models dynamics. I call this the **principle of event-symmetry** ^[4] and it can be applied to models that go beyond random graphs. In the general case the principle says that an event-symmetric model of physics must have invariance under a symmetry which includes a permutation group over spacetime events. Since I introduced this concept as an approach to quantum gravity in 1994, I and a number of other researchers have independently worked on the idea with interesting preliminary results ^[4-13].

Phases of spacetime

We are all familiar with different **phases** of matter – liquid, solid, gas, plasma, quark soups and many more possibilities at much higher temperatures and densities – but what about spacetime, can it exhibit phases too? Different phases are often distinguished by their symmetries. When a liquid freezes it forms a crystal structure characterised by its discrete rotation and translation symmetries. In particle physics matter at high energies has a larger gauge symmetry that is broken by the Higgs mechanism in the low energy phase to the smaller symmetry of the standard model. We are not accustomed to seeing spacetime undergo phase transitions. It does not freeze at low temperatures

to a form that breaks Lorentz invariance, but in an event-symmetric spacetime the permutation symmetry is larger than the diffeomorphism invariance of general relativity. We can expect that at sufficiently high temperatures spacetime itself will undergo a vaporization-like phase transition to a fully event-symmetric phase where its geometry and topological structure is lost ^[6]. Even time and causality would cease to have meaning in such a state. This phase should characterise the conditions of the spacetime singularity in a black-hole or at the very start of the big bang.

It is a difficult task to define an event-symmetric random graph model that exhibits the right phases of spacetime. Recent work on a class of models known as **Quantum Graphity** ^[13] has shown that it is possible to have a low temperature phase with a structure like a low-dimensional solid crystal. In the analogy with the phases of matter, spacetime as we know it behaves more like a liquid. Nobody knows if a random graph model can pass through such a phase. We would need this phase to persist at low temperatures with no crystalline phase to get a realistic model of spacetime. In my opinion we need to abandon random graphs in favour of random matrices to accomplish this goal.

Random Matrices

The configuration of a random graph can be represented using an **adjacency matrix** A with components a_{ij} which are equal to one if the nodes i and j are connected in the graph, and zero otherwise. So a random graph model is also a **random matrix** model. In quantum mechanics logical certainties become complex valued amplitudes. In a random matrix model it is equally possible to replace the zeros and ones of the adjacency matrix with complex valued components. Permutations of the space time events is equivalent to a similarity transformation using a permutation matrix P

$$A \rightarrow P^{-1}AP$$

A permutation matrix is a special case of an orthogonal or unitary matrix. When making the transition from random graphs to random matrices it is natural to extend the permutation symmetry to the fuller matrix group. The action $S(A)$ in the Lagrangian can then only depend on quantities invariant under unitary transformation, i.e traces of powers of the matrix.

Random matrix models have been a subject of study by mathematicians and physicists for many years and much is known about them. A single matrix model like the one we developed from a random graph can be solved by diagonalising the matrix and looking at the distribution of eigenvalues. This is interesting but not sufficiently rich to exhibit anything like the emergence of spacetime.

More advanced matrix models have used more than one matrix. A common trick is to define a matrix X^a for each co-ordinate direction you want to reproduce in your spacetime. The action then contains commutator terms such as

$$S_0 = \sum_{a \neq b} [X^a, X^b]^2$$

In the classical limit this term goes to zero forcing the matrices to commute. They can then be simultaneously diagonalised and the corresponding eigenvalues across the matrixes are interpreted as the classical co-ordinates at the point. In the quantum theory the matrices only approximately

commute and the off-diagonal terms contain gauge field components. Models of this type have been successfully used in QCD and other gauge theories, but are they suitable for quantum gravity?

As with random graphs, the spacetime that emerges from these random matrix models of gauge theories is flat and solid. Can there be a natural process where a more liquid-like curved spacetime is dynamically generated from an event-symmetric model? In fact nature provides an affirmative answer in the form of soap bubbles. Liquids at the atomic scale are molecules with interaction forces that are invariant under permutations of the molecules, yet under the right circumstances they spontaneously form the two-dimensional curved surfaces of bubbles. This is perfectly analogous to what we seek from an event-symmetric model of curved spacetime.

Event-Symmetric String Theory

The leading candidate for a theory of quantum gravity is **superstring theory**. Event-symmetry is not a theory of quantum gravity itself. It is a symmetry principle that can be applied to theories of quantum gravity including string theory.

It has long been said that the true foundations of string theory are unknown. The formulations we have are all either perturbative or limited in scope. It is known that different versions of string theory are connected by a web of dualities but we lack a unified mathematical structure that encompasses all of its forms in one.

It would be natural to try to explain the underlying structure of string theory using principles of symmetry. This would put it on an equal footing with general relativity and gauge theories that have been successful in the past. Yet the search for the symmetries of string theory has been abandoned by string theorists some time ago. It has been said that dualities in string theory show that symmetries are not of fundamental importance because different dual formulations have different symmetries. Yet the dualities themselves contain groups of self dual structures. The challenge is to understand why the unknown overall symmetry breaks in different ways.

Holographic Principle

The thermodynamics of black holes presents an information paradox discovered by Stephen Hawking. Using classical reasoning, information that falls into a black hole should be lost to an outside observer, but this contradicts the principles of quantum mechanics. Information conservation is required by unitarity of the S-matrix. Hawking's paradox was eventually resolved using a **holographic principle** that limits the amount of information inside a black hole to the number of bits that can be displayed on its event horizon, one per Planck area. This limit allows for a complementarity principle that means information that descends into a black hole with an internal observer can also reappear in the form of Hawking radiation for an outside observer. There is no classical description of such a dual process, just as there is no classical solution for a black hole that time reverses to form a white hole. Such phenomena are strictly quantum mechanical. After it was shown through AdS/CFT duality that a holographic principle is possible in string theory, it was finally accepted that this was a valid solution to the black hole information paradox.

The holographic principle suggests a huge reduction in the number of observable degrees of freedom in string theory. I would argue that this is only possible if there is a huge hidden symmetry like event-symmetry.

Witten's Puzzle.

Another argument against symmetry in string theory was presented by Witten in 1993^[14]. If string theory includes a universal symmetry then it should unify all gauge fields and the diffeomorphism group of spacetime into one group. But string theory exhibits topology change and dualities which encompass different manifold structures. The diffeomorphism groups on topologically different manifold are also different, so the universal group of string theory would have to contain them all.

Witten could not explain how different diffeomorphism groups could be contained within one group acting on the same set of spacetime events, but any diffeomorphism group is a subgroup of the group of permutations on the set of events in the manifold. Only by using event-symmetry can we ever hope to unify the symmetries of nature in string theory.

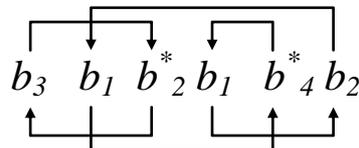
M(atrix)-Theory

In 1997 string theorists at Stanford provided a M(atrix) model for **M-theory** in which space-time is an emergent structure^[15]. Initially the matrices were considered as a large gauge transformation on a set of **D-instantons** whose co-ordinates in the classical limit are given by the **eigenvalues** of the matrices. A reinterpretation was offered later^[16] where the eigenvalues were reinterpreted as space-time events with diffeomorphism invariance embedded as permutation groups in the matrix symmetry. Despite this acceptance of event-symmetry into the fundamentals of string theory few have investigated its implications.

In the subsequent development of **matrix string theory** as originally developed by Lubos Motl^[17], the 11 dimensional M-theory is reduced to 10 dimensions and strings emerge by associating permutations with the eigenvalues. The permutations can be decomposed into cycles which are the origin of the strings. Motl called this "**screwing strings to matrices**". In event-symmetric string field theory^[18-22] we use a quantised version of this process that might be called "screwing strings to creation and annihilation operators".

Event-Symmetric String Field Theory

An algebra for **event-symmetric string field theory** is generated from a set of fermionic creation and annihilation operators b_i, b^*_i where i indexes spacetime events, together with a permutation σ acting on an n -tuple of generators. The permutation σ can be decomposed into cycles which can be represented graphically as arrows connecting the operators. A typical example of a generator of the string algebra looks like this



Multiplication in the algebra is defined by juxtaposition of these diagrams modulo relationships derived from the commutation relations of the underlying operators e.g.

$$\begin{array}{c} \uparrow \quad \uparrow \\ b_i^* \quad b_j \\ \downarrow \quad \downarrow \\ \diagdown \quad \diagup \end{array} + \begin{array}{c} \uparrow \quad \uparrow \\ \diagdown \quad \diagup \\ b_j \quad b_i^* \\ \uparrow \quad \uparrow \end{array} = 2\delta_{ij} \begin{array}{c} \uparrow \quad \uparrow \\ \downarrow \quad \downarrow \end{array}$$

This defines a Z_2 graded necklace algebra from which a supersymmetric **necklace lie algebra** ^[23] can be obtained using the associated commutation and anti-commutation relations. In event-symmetric string field theory this superlie algebra is the symmetry of the theory ^[22].

Foundational Implications

Hidden symmetry

The world around us is full of symmetry, but much of it is hidden until you look carefully. As we walk around in the open we notice that things are held down by the force of gravity. It makes the third dimension of upwards motion seem different from the other two in which we can move more freely. In the past people thought this distinction was fundamental and must continue beyond the horizon for ever. They believed in a flat Earth. They also noticed that objects have a tendency to come to rest on the ground and again they thought it was a fundamental law of physics. They also believed that the Earth was the centre of the world with very different rules governing motion in the heavens.

These notions were dispelled by the observations of Copernicus, Galileo, Newton and finally Einstein. They have been replaced with principles of symmetry. The Earth is round and the laws of physics work equally in all directions. It was just the local appearance of gravity that made it seem otherwise. The Earth is no centre for the laws of physics which actually work the same everywhere throughout the universe. Even motion is relative and we do not notice the passage of the Earth through space. All these ideas are unified in the symmetry principles of relativity.

In general relativity the invariance of the laws of physics extends to accelerated motion and general co-ordinate transformations of space. It is as if we should be able to distort space and time like sift rubber foam and not notice the consequences. But this symmetry is hidden by the local form gravity takes just as gravity once hid from us that the Earth is curved. If we lived in the vicinity of a black hole the curvature of spacetime would be more noticeable and our physical intuition could be very different.

Our experience also tells us that time proceeds in one direction from past to future and not in reverse. The laws of physics at the subatomic level tell us that this rule is not fundamental. Once again it is only a local effect which may not be supported on the largest scales of the universe. Its origins are found in the large cosmological singularity that we know as the big bang. In these extreme corners of spacetime a huge symmetry shapes the universe bringing order to the flow of information. Far beyond the horizon of our observable region in the cosmos there may be other

singularities some of which push the flow of time in the opposite sense. Temporal causality as an explanation of our existence must be abandoned

Hidden event-symmetry

The discovery of event-symmetry in the laws of physics means that even the continuity of space and time must be forgotten. These features of our world are merely dynamical properties valid in the part of the universe where we live. Just as the local background state of gravity hides the symmetry of general relativity from our normal experience, so the background state of string theory hides the event symmetry from us. The symmetry tells us that we can cut up space and time and rearrange the pieces in any permutation we please. The laws of physics would correctly describe the new random sequence of events. If we could experience life within the singularity of a black hole this would be apparent to us, but in the coldness of space it is hidden.

Event-symmetry is just one part of a huge gauge symmetry that underlies string theory. If we can understand it fully we should be able to reformulate string theory in the absence of space and time. Different dual descriptions in different dimensions will be just different configurations of emergent spacetime. The huge redundancy of information that the symmetry encompasses will explain how the universe we inhabit appears as a holographic image projected from smaller quantities of information stored in the observable invariants of the symmetry.

To come to terms with such physics we must be prepared to work without the use of temporal causality. It is a hard principle for physicists to give up. Some are prepared to give up all structure on spacetime except causality. They don't go far enough.

The universe must be described by a vast ensemble of events devoid of spacetime structure. These events are linked by a web of possibilities allowed by the huge underlying of symmetry of string theory. Time emerges only because a solution fits in which the Lorentzian signature of spacetime prevails .

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