Some comments of the physical interpretation of Riemann zeta in TGD

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Abstract

The Riemann zeta function ζ and its generalizations are very interesting from the point of view of the TGD inspired physics. $M^8 - H$ duality assumes that rational polynomials define cognitive representations as unique discretizations of space-time regions interpreted in terms of a finite measurement resolution. One implication is that virtual momenta for fermions are algebraic integers in an extension of rationals defined by a rational polynomial P and by Galois confinement integers for the physical states.

In principle, also real analytic functions, with possibly rational coefficients, make sense. The notion of conformal confinement with zeros of ζ interpreted as mass squared values and conformal weights, makes ζ and L-functions as its generalizations physically unique real analytic functions.

If the conjecture stating that the roots of ζ are algebraic numbers is true, the virtual momenta of fermions could be algebraic integers for virtual fermions and integers for the physical states also for ζ. This makes sense if the notions of Galois group and Galois confinement are sensible notions for ζ.

In this article, the properties of ζ and its symmetric variant ξ and their multi-valued inverses are studied. In particular, the question whether ξ might have no finite critical points is raised.

1 Introduction

The basic motivation for studying Riemann zeta function [A1] (https://cutt.ly/oVNSltD) and its generalizations (https://cutt.ly/3VNPYmp) is that ζ and its generalization are very interesting from the point of view of the TGD inspired physics. I have considered a possible strategy for proving Riemann Hypothesis (https://cutt.ly/CVNPKpZ) in [L1] [K1].

In the TGD framework [L2, L3, L6], the roots of rational polynomials P have interpretation as mass squared values and conformal weights. The momentum components of virtual fermions are in an extension of rationals defined by a rational polynomial P determining the space-time region.

The condition that mass squared values are ordinary integers follows from Galois confinement, implying that momentum components are ordinary integers. Conformal confinement is a stronger condition and requires that the total conformal weight of a physical state as the sum of conformal weights for composite virtual fermions vanishes [L7, L4]. Positive and negative conformal weights representing non-tachyonic and tachyonic mass squared values sum up to a vanishing total mass squared value. This condition generalizes the masslessness condition natural for generalized conformal invariance and can be satisfied.

One can consider a generalization of this picture since polynomials could be replaced with real analytic functions, possibly with rational Taylor coefficients.

1. The roots are replaced by zeros of a real analytic function f. The extension of rationals would be defined by the roots of f. For real analytic functions, roots come as complex conjugate pairs of particles with complex conjugate weights are real. There the sum of roots for a many-particle state for generic f can be real but not an integer in general so that Galois confinement in strong form is not possible.

2. Riemann zeta and its generalizations are completely exceptional since the total conformal weight can be an integer. For Riemann zeta there must be 4 particles with non-trivial root as conformal weight per single particle with trivial root as conformal weight.

3. The structure of the graph of the inverse of $\zeta(s)$ or of its symmetric version $\xi(s)$, which does not possess trivial zeros and the pole at $s = 1$, is of considerable interest.

   Conformal confinement is not possible for ξ since only non-tachyonic roots are possible. Whether this can be regarded as a pro or con argument, is not clear. It is now clear that also space-time surfaces with Minkowskian signature of the induced metric and having light-like boundaries, are allowed by the boundary conditions stating that no isometry currents leak out from the interior of the space-time surface [L5], and for these spacetime surfaces, ξ and its generalizations might be of interest. At the light-like 3-surfaces, Chern-Simons Kähler action would contribute to the action and one TGD could be characterized as a kind of geometry-number theory-topology trinity [L4].

   Riemann zeta and its generalization could be more natural for the space-time surfaces without boundary: for them light-like 3-surfaces would serve as boundaries between Minkowskian and Euclidean regions.

4. If ξ has no critical points in a finite complex plane, the structure of the graph of ξ is analogous to an infinite singular covering of the plane such that infinity would represent the point at which the sheets meet. For ζ $s = 1$ and $s = \infty$ represent analogous points.

A possible interpretation is that in TGD, rational polynomials give discrete cognitive representations as approximations for physics. Cognitive representations are in the intersection of p-adicities and reality defined by the intersection of reals and extension of p-adics defined by the algebraic extension of the polynomial P defining a given space-time surface. Continuum theory would represent real numbers as a factor of the adele.

One can ask whether the various zeta functions consistent with the integer spectrum for the conformal weights and possibly also with conformal confinement, appear at the continuum limit and provide representations for the space-time surfaces at this limit? In this framework, it would be natural for the roots of zeta to be algebraic numbers [K1]. Also in the case of ζ, the virtual momenta of fermions would be algebraic integers for virtual fermions and integers for the physical states. This makes sense if the notions of Galois group and Galois confinement are sensible for ζ.

The notion of ζ generalizes. The so-called global L-functions (`https://cutt.ly/3VNPYmp`) are formally similar to ζ and the extended Riemann hypothesis could be true for them. Algebraic integers for a finite extension of rationals replace integers in the ordinary ζ and one has an entire hierarchy of L-functions. Could one think that the global L-functions could define preferred extremals at the continuum limit?

In this article, the properties of ζ and its symmetric variant ξ and their multi-valued inverses are studied. In particular, the question whether ξ might have no finite critical points is raised.

2 Basic facts about Riemann Zeta

Consider first some basic facts about Riemann Zeta (see `https://cutt.ly/yVLm0JG` and `https://cutt.ly/YVLm7oa`). The basic definition of zeta function is given by the following equation

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{x^{s-1}}{e^x - 1} \mathrm{d}x \quad ,$$

where

$$\Gamma(s) = \int_0^\infty x^{s-1}\, e^{-x}\, \mathrm{d}x$$

is gamma function generalizing the factorial "n!" to $z!$ and satisfying $\Gamma(z+1) = z\Gamma(z-1)$.

The Weierstrass representation for the gamma function is given bt

$$\Gamma(z) = \frac{e^{-\gamma z}}{z} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right)^{-1} e^{z/n} \quad ,$$

where $\gamma \approx 0.577216$ is the Euler–Mascheroni constant. There is a temptation to separate the exponentials $e^{z/n}$ to a separate product $e^{(\sum_n 1/n)z}$. The sum is however equal to $\zeta(1)$ and diverges.

ζ satisfies the functional equation

$$\zeta(s) = 2^s \pi^{s-1}\ \sin\left(\frac{\pi s}{2}\right)\ \Gamma(1-s)\ \zeta(1-s) \quad .$$

The symmetric variant of zeta function called ξ function is defined as

$$\xi(s) = \frac{1}{2}\pi^{-\frac{s}{2}} s(s-1)\Gamma\left(\frac{s}{2}\right)\zeta(s) \quad ,$$

and satisfies

$$\xi(s) = \xi(1-s) \quad .$$

.

One can write ξ as a product of two terms. An exponential decaying exponentially in x-direction and modulated in y-direction in an oscillatory manner and a term proportional to what might be called reduced gamma function $\gamma(s/2)$:

$$\xi(s) = e^{-Ks}(s-1)\gamma(s/2)\zeta(s) \quad ,$$

$$K = (log_e(\pi) + \gamma)/2 \simeq 1.012257 \quad ,$$

$$\gamma(s) = \prod_{n=1}^{\infty} \left(1 + \frac{s}{n}\right)^{-1} e^{s/n} \quad .$$

$\gamma(s)$ converges since it is the product of factors $exp(s/n)$ divided by its first order Taylor polynomial. From this expression it is clear that $\xi(s)$ does not have trivial zeros of ζ nor its pole at $s = 1$.

3 What could one say about the inverses of ζ and ξ?

The functional composition "$\circ$", in particular iteration of polynomials gives new polynomials. For Riemann zeta the functional iteration would give new analytic functions. The zeros of $\zeta \circ \zeta$ would be inverse images of the zeros of zeta by ζ^{-1}. These zeros do not have half-odd integer real parts so that the iteration of ζ does not look like a physially plausible idea.

Despite this, one can try to build a view of the graph of ζ^{-1} in C^2 with coordinates ($w = u + iv, s = x + iy$).

3.1 The behavior of ζ at $s = \infty$

How does ζ behave at $S = \infty$? The following proposal is an outcome of my naive amateurish thinking.

1. Zeta has an infinite number of zeros: the non-trivial zeros at the critical line $x = 1/2$ and trivial zeros at $x = -2m$, $m > 0$ integer.

   From this it is easy to believe that ζ maps an infinite number of finite points to a finite given point. The point ∞ is an exception: only the point $s = 1$ is mapped to it since ζ has only a single pole.

2. One can say that ζ approaches zero at $s = \infty$. How? For polynomials the number of zeros minus poles defines the winding number. ζ has a pole of multiplicity one at $s = 1$. The winding number with respect to $s = 1$ is given by the integral of $(1/2\pi)\oint_C dln(\zeta)$ over a closed contour C circling all zeros, including also $s = \infty$. This contour encloses $s = 1$ so that the integral equals to $n = 1$. Hence the non-trivial and trivial poles and the circle at infinity give infinite contributions of opposite sign summing to $n = 1$.

   At infinity ζ should behave like a pole of infinite order that is $1/(z-s)^n$, $n \to \infty$, $s \to \infty$. The situation is easier to describe using the Riemann sphere as a compactification.

3.2 Critical points of ζ

The graphs of ζ and ζ^{-1} are minimal surfaces by the fact that the Kähler metric appearing in minimal surface equations can be replaced with flat metric. It would be interesting to understand what kind of minimal surface is associated with ζ^{-1}. The periodic trivial zeros suggest a periodic lattice-like structure. Same question can be made for ξ.

1. Especially interesting are the points $d\zeta/ds = 0$ since for these the derivative of ζ^{-1} diverges. This means that two branches of the graph of ζ^{-1} meet at these points. A simple example is the graph of the inverse function of $sin(\pi x/2)$. This function has critical points at $x = (2n+1)\pi/2$ with a vanishing derivative. By turning the graph of this function by $\pi/2$, one obtains the graph of the inverse function as a many-valued function.

   By the functional equation, ζ is proportional to $sin(\pi s/2)$ for $x < 0$, which gives rise to the trivial zeros at $x = -2n$. ζ is real along negative real axis and its derivative vanishes at some point between two subsequent zeros of $sin(\pi s/2)$. Reality implies that one has $\partial_y v = \partial_x u = 0$ and $\partial_y u = -\partial_x v = 0$ implies $d\zeta/ds = 0$ holds true at these points. These points are saddle points for both u and v (minima or maxima are impossible by analyticity). Hence one the geometric structure of the graph of ζ^{-1} is similar to that for $sin(\pi s/2)$.

2. What about the critical points possibly associated with the non-trivial zeros of ζ? It is known that they are have multiplicity 1 so that $d\zeta/ds$ is nonvanishing at them. Also it is known that there are no critical points with the critical strip (`https://cutt.ly/jVLQaNj`.

One can assign to the zeros of ζ branches by solving the zeros $\zeta(s) - C$ in the environment of given zeros. Allowing C to run over complex numbers around each zero, one obtains all branches as a union of the orbit zeros of ζ as C is varied. Some of these branches are expected to meet each other and fuse together. The branches associated with the inverse of a polynomial of real variable give an idea of what would happened

1. The intuitive idea that ζ is analogous to a polynomial of infinite degree and the fact that polynomials have critical points between their roots suggests that the number of critical points is infinite and they correspond to pairs of zeros of ζ. The existence of a pole of infinite order at $s = \infty$ can however change the situation.

   The branches of ζ^{-1} could meet at the critical points in the region $x > 1$ and also in the region $x < 0$.

2. The simplest option would be that there are no critical points except those associated with trivial zeros. This would mean that copies of the complex plane associated with non-trivial zeros of zeta meet at infinity.

   Unfortunately, this simple scenario cannot be true. It is known that ζ has critical points in the region $x > 1$ (`https://cutt.ly/cVLQxrZ`). For the points considered, the value of the modulus of zeta is above a lower bound so that they do not give rise to zeros which would be critical points. The Riemann hypothesis of course implies this.

3.3 Does ξ have critical points?

The symmetric analog $\xi(s)$ of ζ does not have trivial zeros, and the corresponding critical points for $x < 0$. Also the pole at $s = 1$ is missing. The contour enclosing the critical points gives an infinite contribution to winding numbers as the number of non-trivial zeros and the contribution from the infinite circle around $s = \infty$ cancels this.

If no new critical points emerge in the critical stripe, the critical points of $\xi(s)$ are symmetrically located in regions $x > 1$ and $x < 0$. The deformations of critical points of ζ outside the critical stripe need not give rise to critical points of ξ.

Consider now the conjecture that ξ has no critical points, which correspond to turning points for ξ^{-1} at which its branches meet.

1. The conjecture is false for ζ because it has critical points associated with trivial zeros and is also known to have critical points in the region $Re(s) > 1$ (at least).

2. It is not obvious that the conjecture is untrue for $\xi(s)$. The branches of ξ^{-1} meet at the critical points. Does the absence of finite critical points imply that the the 2-D branches of ξ^{-1} as copies of complex plane intersect at $w = 0$, which corresponds to the limit $\xi(s \to \infty)$?

   Could one say that $w = 0$ is a critical point of ζ including not only the non-trivial zeros of ζ with multiplicity one but also $s = \infty$ with infinite multiplicity? Criticality in turn would imply that an infinite-fold turning point for ξ^{-1} is in question so that $\xi^{-1}(w = 0)$ has divergent derivatives for $s = \infty$. All derivatives of ξ would vanish at the limit $s \to \infty$. This conclision is consistent with the outcome of winding number based arguments suggesting a pole of infinite order at $s = \infty$.

3. For ζ, the situation is the same as for ξ at $w = 0$. At $w = \infty$ the situation is different since ζ has a pole at $s = 1$. The formula for the inverse $\zeta^{-1}(w = \infty) = 1$ implies the vanishing of all its derivatives at $w = \infty$. $w = \infty$ and $w = 0$ would be infinitely critical points of ζ.

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